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Contextual utility framework
Application in portfolio diversification theory

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Abstract

Purpose: Arrow-Debreu's seminal works generated the field of portfolio diversification theory, followed by seminal works of CAPM and allied models. However, it has been observed since past decades that CAPM and allied frame works have not predicted well the investment behaviors under contexts like uncertainty. Recently quantum-like modelling or quantum decision theory has been formulated to account for choice behavior under contexts, which is a comprehensive decision-making framework based on quantum logic rather than standard Boolean logic. This short paper provides a possible set up for portfolio choice behavior under contexts based upon the very frame work.

Design/ Methodology/ Approach: model proposed is based on the Hilbert space modelling framework of quantum decision theory. Some parallels with Ellsberg paradox in decision theory is emphasized.

Findings: This paper is a theoretical model, which aims in overcoming limitations of neoclassical diversification theory by improving the descriptive and predictive power of the theory. Overall, the current paper can be positioned within the Econophysical and Quantum-like modelling paradigms which are attempts to build a more comprehensive Economic theory.

Research limitations: The author accepts that the theoretical model proposed here needs to be tested, suitable data analysis would be the next immediate step to follow.

Originality/ Value: one, to propose a novel application of quantum decision theory/ framework to a pressing real-world problem of portfolio diversification, which is crucial for investors, two, also to raise awareness among main stream practitioners of economic theory about the emerging field of quantum-like modelling in economics and finance.

Key words: quantum decision theory, quantum-like modelling, Arrow-Debreu, risky assets, diversification, Born's rule, Hilbert space modelling

1. Introduction

Kenneth Arrow, is one of the founding fathers of modern neoclassical economics. Arrow's contribution and legacy runs through generations, and many iconic economists like Professor Amartya Sen have been influenced by Arrow's seminal contributions in different areas, for example: social choice theory, organisations, diversification theory to name a few.

The current paper is concerned with one such area which Arrow and Debreu (1954) developed, and which then ushered vigorously (1989) generating Nobel prize winning theories, for example CAPITAL ASSET PRICING MODEL, CAPM. Portfolio diversification literature, the theory being referred to here, is however a versatile area with a very dense history. Standard utility optimising frameworks to behavioural models (Dhami, 2017) are concerned with the area. In the current paper we are not deviating totally from the utility framework, but rather providing a more robust basis by introducing 'contextual utility' frame work, which has been recently proposed (Aerts et al, 2018).

For the last few years there has been a surge in interest in quantum like[1] modelling in social science, specifically in decision making models (Khrennikov and Haven, 2013)[2]. The main reason for such an upsurge is the failure of classical decision theory (based on Set theory and Boolean Logic)[3] to explain various facets of human behaviour in general, for example behaviour of decision makers under ambiguity, which further give rise to some deviations or anomalies hard to explain from the basis of classical set theory based decision theory. Some of the examples of such anomalies being, order effects (which in probabilistic terms mean P(A and B) NOT = P(Band A), for two events A and B which may not be correlated, such inequality is not possible according to standard set theory based probability discourse), conjunction and disjunction fallacies (again giving rise to anomalies in probability rules), failure of the basic law of total probability[4], failure of standard Bayesian probability theory to explain the updating of beliefs under uncertainty since the Bayesian theory does not allow updating from 0 or 1 prior to a further significant posterior probability, also known as the zero prior trap, this law actually does not hold in reality for example in financial markets where there can be abrupt jumps in beliefs due to uncertain environment, and many alike[5]. Pioneers in this emerging field (Bagarello et al, 2015) have noted with surprise and also with caution that a new

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[1] Quantum-like since we are by no means claiming any physical quantum theory underlying agents choice behavior, rather adopting the mathematical foundations.

[2] There is certainly another parallel movement, econo-physics, which mainly borrows from the statistical mechanics and apply such foundations and concepts to price and return distributions in financial market states, and allied areas.

[3] Kolgomorovian measure theory (since 1933).

[4] For details we can look into Khrennikov and Haven (2013,2009)

[5] For a detailed exploration of the topic we can consult Khrennikov and Haven (2013).

foundation based on Hilbert space set up and Non-Boolean logic is called for in decision making theory for resolving such anomalies.[6]

In basic quantum theory a state is supposed to live in a finite or infinite dimensional complex Hilbert space. Hilbert space is a complex vector space which has a defined norm on it, and is spanned by basis vectors. Such basis vectors are orthogonal to each other. Though linearity property holds on the space, there are some logical operations which does not hold, namely, commutation and distributive properties do not hold, which can further be used to explain anomalies such as order effects observed in human decision behaviours if such a foundation is adapted[7].

Unlike in whole of the classical decision theory Hilbert space description of an initial pure state is that of a linear superposition of basis states, such a superposition can never be compared with any probability distribution over states but simply a superposition or possibility of all states co-existing till any measurement is done. The coefficients of basis states in a superposition description gives the probability amplitudes, squared moduli of whom provides the probability of such states getting actualized when a measurement is performed[8].

Hence in case of decision theory modelling, or cognitive modelling a pure state or a pure belief or mental state is considered to be represented by such Hilbert space model, where the space need not to be infinite dimensional and complex. Density matrix representation can be given to ensemble of such pure mental states. Here we always note that the main purpose of such description is to measure the probability to reduce to one of the Eigen values from the superposed state, which is obtained by the famous Born's rule of squaring the amplitudes as in the superposition description (Basieva and Khrennikov, 2017).

It is important to note that quantum theory is an inherently probabilistic theory, which means there is a natural limit to predict or measure observables with certainty. Unlike classical decision theory then quantum decision theory is also based on an irreducible randomness. This point is the fundamental departure from the classical deterministic philosophy where probability can come only due to ignorance.

Recently there has been an upsurge in studies applying Quantum probabilistic formulation outside physics, mainly in various areas of decision making (Bagarello, 2015). The novel movement which has thus begun created a new branch of knowledge itself, the so called QDT or quantum decision theory. The application of the very theory

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[6] Caution since as noted by Khrennikov and Haven (2013) there can be some further complex interference effects in human decision making (hyperbolic interference terms) which may be difficult from the quantum modeling perspective to explain.

[7] Several good introduction in quantum modeling / Hilbert space based modeling can be availed, for example, Neumann (1932), Susskind (2014), Lancaster and Blundell (2014) and others.

[8] This is the central issue of measurement problem in quantum theory, and there are many interpretations of the same: collapse of the wave function or Copenhagen interpretation being the orthodox interpretation which demands that once the initial state is left alone it evolves under the unitary evolution rules, however once any observation is made there is a random collapse of the initial superposed state into one of its Eigen values, whose probability is the square of amplitude rule.

in the areas of cognitive and mathematical psychology is pioneered by Busemeyer and others (Bruza et al, 2015), whereas the application of the same in the areas of management, finance and social science in general is pioneered by Khrennikov and Haven (2013), along with many other notable contributions by Yukalov and Sornette (2011).

The basic reason for application of the very set up is that as has been noted by the psychologists since Kahnemann (1992) the human decision making in general can not be captured by the standard EUT (expected utility theory) which is again grounded on the classical Kolmogorovian set theoretic / measure theory of probability(Kolgomorov, 1933). there are many violations of the basic axioms of EUT as proposed by founders of the same theory like Savage (as in Khrennikov and Haven, 2013, 2009) and others, for example violation of sure thing principle, presence of conjunction and disjunction effects in human decision making, presence of order effect in human cognition, and over all the inability of the standard probabilistic theories to explain human decision making under uncertainty or ambiguity.

Yearseley (2017) have observed in many studies that violation of the classical probabilistic theoretic predictions is mainly due to contextual behaviour of decision makers, different contexts under which the same decision maker acts make it difficult to represent the behaviour based on standard measure theory. Khrennikov even goes further to formulate a specific version of quantum decision theory, now called as the Vaxjo interpretation[9], which attempts to explain the so called interference or non-classical results of quantum probabilistic predictions based on contextual probabilities. More on the same is elaborated below in the relevant literature sections.

Bruza et al (2015) have further observed that managerial/ organizational decision making carries all the traits which warrants the use of QP theory rather than standard EUT for faithful description as well as predictions. Uncertainty, contextuality, ambiguity, and various types of violations of predictions of EUT are the hall mark of decision makings in an organizational set up. There have been very recently some efforts to describe organizational decision making based on the QDT (Khrennikov and Haven, 2009), and such studies are gaining currency rapidly. A new paradigm is on rise.

For economic decision making theory it is very critical to consider the limitations of the expected utility theory (EUT), since the Neoclassical modelling fundamentally is based on the premises of EUT. EUT has been since the works of Savage and others the standard utility modelling. This theory is so much successful in general that the whole of neoclassical economic modelling is based on the applicability of the EUT. However, deep down the very theory is based on the deep axioms of classical probability theory.

One specific axiom is the so called sure thing principle. Savage originally formalized the principle, and the whole EUT is based on the very principle. Sure thing principle is actually quite simple to follow, and the crux of the same is similar to the irrelevance irrelevant options while making decisions. For example, if Bob is asked whether he will buy a house if the presidential candidate A wins, and the answer is yes, and the answer

[9] Where the main technique deployed is the use of POVM or positive operators with non-negative real Eigen values, which helps describe various effects like order effects. This model also accounts for the interference terms in decision making outcomes.

remains yes even if candidate B wins, then Bob is indifferent to the both the candidates' win, and hence if he does not have any information on the win he should till choose buy option.

However there is a strong evidence since Kahnnemann (1992) that sure thing principle (SUP) is regularly violated under uncertain contexts, for example if Bob is under an ambiguity scenario regarding the wins then he might behave not according to the sure thing principle at all. The same violation of SUP is also observed in case of experimental data in prisoners' dilemma scenarios, the famous game theory scenario of cooperation failure (Giloboa et al, 2008). In case of prisoners dilemma situation there is a clear dominant strategy for each player, which they should play irrespective of what the other player choose. However there is a very strong experimental evidence that under conditions of ambiguity or uncertainty the behaviour of players may not coincide with the prescribed Nash equilibrium.

It is quite recent though when the quantum theoretic set up is being used to describe certain economic or financial decision making (Haven and Khrennikov, 2013, 2009). Financial decision making in general is a good candidate for quantum modelling since the information environment is generally uncertain, or ambiguous. Earlier Haven (2003) has described asset pricing model in terms of quantum information theory set up where such a Qubit set up actually can describe the uncertain information environment. Classical or Neo-classical decision theory has failed to capture the uncertain information environment in finance, since the typical probability distribution set up can depict risky scenarios but not uncertain scenarios (Sozzo and Haven, 2017).

Given the backdrop of successful application of quantum like modeling in economics an allied area we can try apply the same foundation to another central area of modern economics: portfolio theory. Since the works of Markowitz (1991), Sharpe (1977), Arrow and Debrue (1954) portfolio theory has been the central tenet of neoclassical economics and finance[10]. However, here too the problem is with describing the behaviour of agents under ambiguity in a faithful way so that the model can capture the real behaviour as closely as possible. There are serious limitations in the theoretical models in the standard frame works, for example in the celebrated CAPM or capital asset pricing model the results are based on the assumption of homogenous expectations of agents. Homogenous expectations or beliefs can only hold in a certain information environment, where at least the probability distributions over future events are a common knowledge. However such an utopian world breaks down under ambiguity or uncertainty about world states. In standard Neoclassical finance models there have a debate on the impact of uncertainty in information environment on the asset prices, but the debate has always been without any consensus, mainly on the exact nature of impact of uncertainty on prices (Miller, 1977 was the first author to model divergence of investors opinions as a measure of uncertainty).

Certainly the prospect theory (Kahneman et al, opcit) and other behavioural models by economists like Thaler, Shliefer, Vishney, and Shiller have captured some important

[10] There have been related models based on information asymmetry also, please check the end ref list, like Stiglitz, Bhattacharya and others.

deviations in so called rational behaviour as the standard utility model would claim. However to formulate such models different behavioural biases, heuristics, complicated utility frame works, Bayesian learning models etc have been used profusely. There seems to be a lack of coherence among all such various modelling approaches.

Contextuality in decision making has been studied in details recently using quantum-like framework (Khrennikov and Haven, 2020), Copenhagen school (or using Leifer's (2016)term ' Copenhagenish' since there are intriguing differences between many sub schools, for example QBISM and Relational quantum mechanics though stem from the Bohr's philosophy of quantum mechanics, but the way they theorise observer and agency are fundamentally different), of QM have held that contexts in which observations are made directly influence the outcomes. Contexts are also equally significant for decision making processes. Certainly there is a huge behavioural finance literature (for example Dhami, 2017)which do base decision making on contexts like uncertainty and ambiguity, but again such models are either based on heuristics or modifications of linear utility frameworks of neoclassical economics. Here one can observe that quantum-like framework might provide a true alternative, since it can use widely used set ups like CHSH inequalities for 'measuring' or 'testing' contextuality. Here we might also observe that 'context' can be defined in different ways, Bohr originally thought of an 'inseparable' whole between the quantum system and the classical measurement apparatuses forming a context, later for example in relational QM context is any general interaction between systems, in QBISM context is provided by the specific 'experiences' which agents have while updating their belief states.

Another fruitful related framework is contextuality by default framework, CbD , whose results coincides with the quantum-like framework for human cognition. CbD takes a statistical view of modelling contexts, as tuples of random variables to be measured together, hence the investigation is on whether they might for a joint probability distribution. QBISM has also been proposed as a general decision making framework, which is personal and subjective use of quantum framework for predicting outcomes and updating beliefs, here too contexts do play central role.

Again as we can see in the current proposal (also see very recently developed proposals by Sozzo (op cit)) the standard neo-classical utility framework it-self can be given a contextual reformulation. Here is where quantum like modelling can provide a more coherent and comprehensive base to observed behaviours under market uncertainties. The current proposal is not suggesting that we need to reject neoclassical optimization framework altogether, since we would like to maintain the underlying economic logic of 'revealed preference' in the spirit of Arrow-Debrue framework, but certainly make the same more robust by introducing contextual decision making, which is at best less developed in the standard framework. The application field for the current proposal is portfolio choice behaviour, since that is the central feature of an Arrow-Debrue economy.

Since the model presented will be a prototype model, the tools used would be basic : finite dimensional Hilbert space modelling with standard orthonormal projectors, however in the appendix a more general version is provided or suggested with POVMs (positive operator value measures).

Some specific models which have attempted to describe decision making (hence asset pricing and portfolio diversification) under ambiguity is discussed briefly below, which might help to distinguish them from the standard neo-classical models.

Epstien's framework (2001, 2007): Epstein and Scheneider (2007) proposed a comprehensive framework of learning by Baysian rational agents in the scenario of ambiguity, which is different from risk. Here again the reference study is that of Ellsberg paradox, which is also the reference for the current proposal. However, one important difference between their framework and quantum-like is that 'fundamental' or 'ontological' uncertainty is described better in the latter. In any Bayesian framework it is the credence of the agents which act as subjective probabilities, which describe the ambiguity atmosphere, however, in quantum-like framework uncertainty is described more deeply by the 'linear superposition' of states allowed in the Hilbert space structure. Such linear superposition of states cannot be reduced to classical mixture of probability states, and thus would describe 'ontological' uncertainty.

Chen and Epstein (2002) however have developed a 'multiple prior' based utility framework which allows agents to exhibit ambiguity aversion. Such a framework is different from the continuous-time stochastic utility models which are based on standard probability assignments by rational agents. However, in real asset markets with deep uncertainty, it has been well documented that ambiguity aversion as well as ambiguity attraction both are exhibited by agents. Recent development of quantum decision theory , which is dynamical in nature demonstrate both aversion and attraction, hence might be considered advantageous.

Based on the above background literature, the paper is organised as follows, modelling background provides the specific approach amongst different quantum-like models which we adopt here, model set-up which provides the basic model with some examples, conclusion and discussion which invites further discussion and thoughts in the same direction.

2. Modelling background

Since late 1990s there is a voluminous literature developed on quantum-like modelling for human decision making. The qualifier 'like' is important here since by no means we propose that there is a quantum physical explanation of human brain (which might be a feasible theory, but here it need not concern us). Overall mathematical description of human decision making is the main aim here, which is generally accomplished by adopting finite or infinite dimensional Hilbert space set up as the state space for describing the decision making process, tools widely used in standard quantum theory framework, viz, self adjoint operators, Neumann- Luder ansatz for state updating or Born's rule for probability computation, are the main ones. In case of human cognition modelling instead of usage of one-dimensional orthogonal projection operators POVM or simply positive operators have been suggested.

Some authors adopt a very experimental view of quantum-like cognition models, we may draw an analogy between quantum state preparation phase and then measurement upon that state to obtain the final result which is 'inherently' probabilistic in nature, with the decision makers state preparation (say the experimental context in which the agent finds herself) and then measurement state which might be agents answering questions related to the choice or the mental state. Such 'measurements' themselves alter the cognitive states of agents, analogous to measurements on quantum systems which update the state of the system as a whole, which can again be mathematically modelled via quantum instruments.

Before describing the basic quantum probability framework more, it's better to point out some cautionary remarks about the limitations of this basic model presented, as below:

First, it is a theoretical model of description of ambiguity aversion (in Ellsberg sense) for the choice over assets or portfolio diversification, with some implications for pricing. Hence the immediate limitation is a lack of thorough simulation type exercise which might be the next step from this proposal. Second, in this discrete finite dimensional Hilbert space treatment continuous-time evolution of behaviour like ambiguity aversion is not exhibited, though in the appendix some suggestions are made. Again 'interference' terms with its implications for ambiguity aversion/ attraction is suggested. Certainly the formula for total probability, or FTP in short, is the main departure of quantum-like modelling from the more familiar measure theoretic probability formulation.

One specific scenario of portfolio diversification might be 'investor's diversity of opinions' which is quite explored in finance, but very recently a quantum-like framework has been used (Khrennikova and Patra, 2019), the current proposal refers to the same.

2.1 Overview of quantum-like modelling set up

Basics of QP (Quantum Probability) framework

We begin with a brief comparison between classical probability theory (CPT) and quantum probability theory (QPT):

The main features of classical probability theory are:

Events are represented by sets, which are subsets of 'Ω. Sample space, sigma algebra, measure (probability)*, are the main features of the related Kolmogorov measure theory.

Boolean logic is the type of compatible logic with CPT, which allows for deductive logic, and basic operations like union and intersection of sets, DeMorgan Laws of set theory are valid.

Conditional probability:$probability(a|b) = \frac{prob(a\&b)}{prob(b)}$; p (b)>0We see conditional probability is a direct consequence of Boolean operations[11]

The main features of Quantum Probability Theory are:

[11] In QM there are different ways of introducing conditional probabilities, either sequential measures, or based on two state vector formalism (Aharnov et al), where expectation values of observables are calculated given final and initial boundary conditions.

State space is a complex linear normed vector space: Hilbert space; Finite/ infinite D, symbolized as H

H is endowed with a scalar product (positive definite), norm, and an orthonormal basis, non-degenerate

Any state can be visualized as a ray in this space

Superposition principle: which states that a general state can be written as a linear superposition of

'Basis states', in information theory language the basis states are $|0>$ or $|1>$.

Measurement: most of the times projection postulate; Measurement implies projection onto a specific Eigen sub-space. Probability, updating can be visualized as sequential projections on Eigen subspaces

Non –Boolean logic is compatible with such state space structure, which means violation of commutation and associative properties.

The main features of Non-Boolean Logic are:

Algebra of events is prescribed by quantum logic. Events form an event ring R, possessing two binary operations, addition and conjunction P (A U B) = P (BUA) (this Boolean logic feature is invariant in Quantum logic). P {A U (BUC)} = P{(AUB)U(AUC)} (associative, property also holds good)

AUA = A (idempotency)

P (A and B) ≠ P (B and A) (non -commutativity, incompatible variables) A and (B UC) ≠ (A and B) U (A and C) (no distributivity)

The fact that distributivity is absent in quantum logic was emphasized by Birkhoff and von Neumann. Suppose there are two events B1 and B2 that, when combined, form unity, $B1 \cup B2 = 1$. Moreover, B1 and B2 are such that each of them is orthogonal to a nontrivial event $A \neq 0$, hence $A \cap B1 = A \cap B2 = 0$. According to this definition, $A \cap (B1 \cup B2) = A \cap 1 = A$. But if the property of distributivity were true, then one would get $(A \cap B1) \cup (A \cap B2) = 0$. This implies that $A = 0$, which contradicts the assumption that $A \neq 0$.

The main features of Quantum-like Modeling of Belief States are:

Bruza et al [27]: cognitive modelling based on quantum probabilistic frame work, where the main objective is assigning probabilities to events

Space of belief is a finite dimensional Hilbert space H, which is spanned by an appropriate set of basis vectors

Observables are represented by operators (positive operators / Hermitian operators) which need not commute

$[A, B] = AB - BA = 0$

Generally, any initial belief state is represented by density matrix/ operator, outer product of ψ with itself $\rho = |\psi\rangle\langle\psi|$, this is a more effective representation since it captures the ensemble of beliefs

Pure states and mixed states

Mixed states: $\rho = \sum_i w_{i'}|\Psi_i >< \Psi_i|$ hence mixed state is an ensemble of pure states with w's as probability Weights.

Some properties of ρ: ρ $=\rho^{+}$, or it is a Hermitian operator, equal to its transposed complex conjugate, for pure states ρ = ρ^2, (ψ, ρ ψ)>0: positivity, Trace ρ = 1

Measuring the probability of choosing one of the given alternatives, which is represented by the action of an operator on the initial belief state

While making decision superposition state collapses to one single state (can be captured by the Eigen value equation).

Observables in QPT represented by Hermitian operators:

A = its transposed complex conjugate

E (A) = Trace (A ρ), every time measurement is done one of the Eigenvalues of the A is realized.

$A = \sum_i a_i P_i$, Spectral decomposition rule: a's are the Eigen values and P's are the respective projectors which projects the initial state to the Eigen subspace with a specific Trace formula: p(ai)= Trace(Pi ρ).

As soon as the measurement is done the state ρ': Pi ρPi/ Tr(Pi ρ) Simultaneously updating of the agents' belief state

A QUICK REVIEW OF FORMULA FOR TOTAL PROBABILITY / LAW OF TOTAL PROBABILITY (LTP), MODIFIED IN QUANTUM-LIKESET UP

3. Model set-up

In this basic model economy as a whole is thought of being composed of some asset classes, and the ambiguity in the investment scenario is considered as to be Ellsberg type. There is no doubt there can be various ways in which uncertainty can be introduced in a financial market/ decision making model. For example based on recent works of Khrennikov et al (2018) one can introduce generalized Heisenberg-Robertson type inequalities in decision making[12]. However if there is common knowledge about the proportion of assets as in this model, we might expect such ambiguities to be absent.

Say the economy is composed of certain categories of assets, call it S, M, and J[13]. Now if the exact proportions of these securities are a common knowledge then there is no ambiguity, and rational investors can always maximize an objective utility function, generating optimal weights and fair prices of these assets.[14]

However if the exact ratios, or proportions of each type of asset is not known, much like the cases in the Ellsberg pay off matrix, then there will always be some portfolio choices

[12] Compatible or in compatible questions asked to agents or investors in this case.

[13] We can imagine these to be senior, mezzanine and junior claims or securities.

[14] UNDER HOMOGENOUS EXPECTATIONS, as in any asset pricing model like CAPM

with ambiguity, and investors would like to avoid such acts/ choosing such portfolios, and rather would invest in such portfolios which are ambiguity free.[15]

Here the investors would choose based on the subjective utility satisfying rule, which can be derived from the operator formalism and Born's rule formulation as in Aerts, Haven, and Sozzo (2017)

However, from the portfolios chosen based on the subjective/ state specific utility satisfying rule would not be equivalent to optimal portfolio as in the unambiguous case here can also be some other world states, or incomplete world states, for example some with 0 M type or 0 J type assets, correspondingly there would be choices which might be completely state dependent and might not be consistent according to standard model, say CAPM[16]

Overall, if ambiguity aversion persists in economy then there might be skewed distribution of asset holdings with inflationary or deflationary impact on prices

Ellsberg experiment based

3.1 Model

Every agent has a choice, or action of selecting risky assets, s $=(E_1, x_1, E_2, x_2, \ldots)$, where E's are the events of choosing specific assets, and x's are potential pay offs from those assets. Any standard asset pricing model will propose that the pay offs are still a function of both idiosyncratic and systematic or market wide uncertainties.

Hence f is a 2n tuples of events and pay offs. However choosing an asset is itself a composite event, since every choice is related choosing over underlying uncertainties accompanying the specific assets (or idiosyncratic factors). Hence any choice can be modelled as a Tensor product of the event space say A (which is a Hilbert space spanned by basis states say $\{|n_i>\}$ and the uncertainty space say B (which is also a Hilbert space spanned by basis states say, $\{|\alpha_i>\}$.

Hence the projector operators on the composite space corresponding to any act f of choosing any risky asset is given by:

$\boldsymbol{P_i = |\pi_i><\pi_i| \; where \; |\pi_i> = |n_i>|\alpha_i>}$ **(1) Where the tensor product is implied.**

Hence we can measure both the state specific utility and the probability of choosing portfolios.

Say if the initial strategic state of the decision maker DM is represented by the density matrix ρ, then the probability of choosing a specific portfolio is given by the Born's rule,

$p_i = trace\ (\rho P_i), where\ \rho\ is\ the\ density\ matrix\ representation$ (2)

[15] We can have this, for example if individual agents have different information / expectations regarding the mentioned proportions

[16] since then these incomplete portfolios will be what held by the investors

Again following Sornette and Yukolov(2011) p(i) can be decomposed into diagonals and off diagonal terms; $\mathbf{p_i = f(\pi_i) + q(\pi_i)}$[17],[18] the later part q(.) has many interesting properties which can be used to explain attraction or repulsion from a specific portfolio choice. For example, in the current model q(.) may generate aversion or attraction towards a specific ambiguous portfolio.

However for our model the main purpose of decomposing the composite choice probability is to show that given the world state, or the information set available to the DMs choice making is inherently probabilistic in nature.

Given the probabilistic nature of choice, now the task is to formulate the state specific, or context specific[19] utility obtained by the DM. we invoke (Haven and Sozzo, 2017) the action operator $\boldsymbol{F = \sum_i u(x_i)\, P_i}$ **(3)**

Hence state specific utility say, $W_v \boldsymbol{= <v|\sum_i u(x_i) P_i|v> = \sum_i u(x_i)|<\pi_i|v>|^2}$ **(4)**

And certainly all the interesting properties of comparison between state specific utilities from different actions, say f, g, h… will hold.

Specific portfolio formulations and choices under ambiguity

The economy is comprised of three types of assets of different risk categories, say, (S, M, J), generally we can assume S refers to senior class, M refers to Mezzanine class and J refers to junior class. Hence the original strategic belief state of the DM comprises S,M and J. for simplicity we assume that risk free rate of return, and returns from the risk categories, and the underlying cash flows are a common knowledge. In such a world the only task which a rational DM is assigned to is maximizing U(x), which then implies asset prices based on benchmark models like CAPM.[20]

However, whenever we relax the common knowledge assumption, we need to formulate the state specific utility, as provided above, $W_v(.)$. However W(.) is also dependent on the subjective attraction/ repulsion factors as shown above, which is embedded in the $p_i(.)$ terms, and these factors are dependent to some extent on the idiosyncratic nature of the underlying assets.

However in our model the ambiguity is introduced on a marstate wide or aggregate level. One simple way would be to base the problem on famous Ellsberg type context, where the exact proportions of S,M and J in the economy is unknown.[21][22]

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[17]Sornette and Yukolov interpret f(2011) as the objective probability part and q(.) as the subjective probability part. If again we live in a classical world of no information asymmetry and homogenous beliefs then q(.) vanishes, as argued by Sornette and Yukolov.

[18] Detailed mathematical properties of f and q parts can be found in Sornette and Yukolov, generally f is the sum of the diagonal terms ; $f(\pi_i) = \sum b^2\langle n\alpha,\rho ,n\alpha \rangle$, and $q(\pi_i) = \sum b^* b\langle n\alpha,\rho , n\beta \rangle$ the sum of the off diagonal terms

[19] action operator formulation, for the initial choice of portfolios

[20] The same problem can also be perceived from known proportions of these classes of assets, say if the ratios are a common knowledge too then always an optimal diversification will be achieved by the rational agents.

[21] We can think this as a parallel construct to three urn example as discussed in Haven and Sozzo (2017).

[22] Intuitively then suboptimal portfolios will be resulted.

Under this scenario, there can be incomplete strategic states also like a state say, u with 0 M type of assets, or w with 0 J type of assets. Along with this there can be :

1. Ambiguous states say choosing only from asset M or asset J, where W(.) can't be measured
2. Ambiguity free states like choosing only S where W(.) can be measured

Hence in this model, there are two types of uncertainty, idiosyncratic / subjective attraction factors, and systemic uncertainty due to Ellsberg type contexts.

Again since, $\boldsymbol{W_v} = <\boldsymbol{v}|\sum_i \boldsymbol{u}(\boldsymbol{x_i})\boldsymbol{P_i}|\boldsymbol{v}>$**, for** any strategic belief state v, the same can also be decomposed of objective and subjective utility parts, namely:

$$W_v = <\boldsymbol{v}|\sum_i \boldsymbol{u}(\boldsymbol{x_i})\boldsymbol{P_i}|\boldsymbol{v}> = \sum_i \boldsymbol{u}(\boldsymbol{x_i}) * |<\boldsymbol{\pi_i}|\boldsymbol{v}>|^2 = \sum_i \boldsymbol{u}(\boldsymbol{x_i})(\boldsymbol{f}(\boldsymbol{\pi_i}) + \boldsymbol{q}(\boldsymbol{\pi_i})) \quad \textbf{(5)}$$

Hence while selecting between two portfolios of risky assets, the comparison will be between the differences between the f and q values of the respective portfolios. Now q (.) part of the utility or the probability measure for choosing one portfoliois taken as to be the subjective attraction factor of the DM towards that choice/ portfolio.

Here q (.) needs to be interpreted as the reflection of choice under an irreducible uncertainty, only if the world state is completely a common knowledge that q(.)=0, and the utility of choosing one portfolio be totally a classical maximization problem.

Specific example of portfolio selections

Here we can construct a simple Ellsberg pay off matrix with three types of asset classes as in the model, S, M and J, and can then consider say 4 actions, f:

Action f_1: choose S type assets only, implying $F = \sum_i u(x_i)\, P_i$ambiguity free choice

Action f_2: choose J types of assets only, implying ambiguous choice

Action f_3: choose from S and M only, implying ambiguous choice again

Action f_4: choose from M and J only, implying ambiguity free choice

Hence the current model captures two types of uncertainties:

1. The fundamental uncertainty captured in the term q(.), however we can measure the state specific utilities here, W_v
2. We can term this as systemic uncertainty, or systemic ambiguity[23]which generates from the incomplete knowledge of the proportions of asset types in the economy, which renders many portfolio choices ambiguous since W_v' s can not be measured, and thus agents would exhibit ambiguity aversion, or choosing suboptimal portfolios.

3.2 Partial ambiguity resolution: Dynamics and Hamiltonian formulation

[23] Two types of uncertainties can be theorized here, one is the fundamental idiosyncratic uncertainty captured in the tensor product, and the other one is the systemic uncertainty.

There are recent main stream studies which have attempted to model partial resolution of ambiguity in asset allocations in a portfolio. However there is still no consensus on the process through which such resolution happens. Again since the standard paradigm is based on Kolgomorovian measure theory, or the learning or belief updating based on Bayesian probability theory there are some serious constraints, as explained in Basieva et al (2014). For example, if the agents who are Bayesian, starts with non-informative or 0/1 priors on some events then its non-trivial to update such states to significantly different posteriors based on Bayesian updating schema.

Here in the model again we have n types of securities, say for simplicity n=3, S,M and J, these may be senior, mezzanine, and junior securities. Hence as we have already seen there can be Ellsberg type ambiguity, with unknown M and J PROPORTIONS. In such a context, there will be incomplete choices, where agents might try to select incomplete portfolios to avoid ambiguity altogether as above.

However, in the current scenario of our model, ambiguity resolution can be based on sudden jumps of belief states from non-informative priors about say the proportions of asset types in the economy.[24]

Hence to capture such jumps in belief states we invoke the creation and destruction operators, a and a*[25].

Let's consider the agent in an ambiguous state of a superposition of state 0 and state 1, say state A, where state 0 has the meaning that the belief state is that proportion of M securities < proportion of J securities, Which means that if this state is actualized then the price of the portfolio is down, Whereas state 1 means proportion of M> proportion of J securities, meaning that if this state actualizes then price of the portfolio is up. The reason behind this assumption is based on the standard portfolio theory, where greater proportion of riskier assets in the portfolio should be producing greater risk premium which should drive down the asset prices. Hence the current model is not typically a behavioral model with irrational choices; here the source of sub optimal behavior comes from genuine uncertainty in the information atmosphere.

With all the mathematical properties of a and a*, we invoke the operator aa* as the price behavior operator here, such that when the operator aa* operates on the initial ambiguous state A the state collapses to state 1, aa* effectively work as the projector operator for the state state 1.

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[24] Such sudden jumps in belief states again can not be captured by standard Bayesian modeling due to the problem of 'zero prior' trap, which states that if the prior of any event is 1 or 0 then any amount of new information is not capable enough to generate different posteriors.

[25] Such operators are used widely in quantum field theory, have their own commutation relations (anti commutation rather for Fermions) and c* aljebra (Khrennikov and Haven, 2013). Such operators are used to generate excitation states from the ground states, for example creation operator will create a particle from the state I0> and the annihilation operator will destroy a particle from the previous state. Number operator which is a critically important operator in quantum field theory can be constructed from such operators. Number operator is utilized for conservation of physical properties.

Ambiguity aversion- attraction and Diversity of investors' opinion and asset pricing: in finance a strong body of work is devoted to asset pricing given diversity of investors' opinion, where the opinion can be about quality of the asset, or future prospects of cash flows. The standard literature is unclear about the impact of such an ambiguous atmosphere on the asset pricing. For example if the diversity of opinion increases due to information asymmetry problems, for example adverse selection, then 'rational' investors should demand a premium which should be reflected in the depressing of the prices. However there are contrary empirical evidence () where price inflation is observed due to diversity of opinions, hence suggesting rather over optimism among investors regarding fundamentals. Hence the impact is unclear. Certainly, there are cognitive bias based explanations from the behavioral camp, but in the above described quantum-like framework such inflation and depression of asset prices might be better described.

Entropic measures

Ambiguity states in the above model, can also in general be represented by Neumann entropy measures, since it measures the purity of states. A pure state has a minimum value of 0, and a completely mixed state has a maximum value which is reciprocal of the dimensionality of the Hilbert space considered.

POVM in decision models

Recently POVM has been used in cognitive modelling related to describing choice behaviour of agents under uncertainty, this is a very helpful tool in describing agents' behaviour in case of uncertainty in financial markets since many interesting results like order effects can be explained. Authors () point out that positive operators are increasingly used to model decision making since in real life scenarios there can be noise in the decision-making process. Positive operators are a class of projection operators which have more general properties, for example, if E is one positive operator then it can be conceived of as $E = M'M$, where M is a self-adjoint operator and M' is the transpose conjugate of M, such that for all such observations $\sum M'M = I$ where I is the identity operator. Again, M can be given a square matrix representation, for example, if ϵ is the noise in the system then $M = \begin{pmatrix} \sqrt{1-\epsilon} & 0 \\ 0 & \sqrt{\epsilon} \end{pmatrix}$. Noise in the system has an important interpretation in the decision theory literature; for example, say due to some noise in the final choice action, or due to some error, the agent rather choosing the optimal chooses a wrong option, now such actions can be represented by positive operators, rather than more stringent projection operators as described earlier. There are several interesting properties of positive operators (Yearsley, 2017), such as: they are non -repeatable (E2 is not equal to E), they are not unique, they are used when the basic elements in the Hilbert space of the model need not be orthogonal, they are used when there are more responses than there are basis states, this last property can be used in the decision making models with noise in the system. Hence, A positive operator valued measure (POVM) is a family of positive operators $\{M_j\}$ such that Pm j=1 $\sum M_j = I$, where I is the unit operator. It is convenient to use the following representation of POVMs: $M_j = V^*_j V_j$, where $V_j: H \rightarrow H$ are linear operators. A POVM can be considered as a random observable. Take any set of labels $\alpha_1,...,\alpha_m$, e.g., for $m = 2, \alpha_1 = yes, \alpha_2 = no$. Then the corresponding observable takes these values (for systems in the state ρ) with the probabilities $p(\alpha_j) \equiv p_\rho(\alpha_j) = Tr\rho M_j = TrV_j\rho V^*_j$. We are also interested in the post-measurement states. Let the state ρ was given, a generalized observable was measured and the value α_j was obtained. Then the output state after this measurement has the form $\rho_j = V_j\rho V^*_j / (TrV_j\rho V^*_j)$. Hence, we see that the agents may still update the belief states following the same Born's rule.

4 Conclusion and further discussion

Recently various concepts from quantum theory (whether the first quantization or quantum mechanics or the second quantization or quantum field theory) has been applied to economic theories, for example, Orrel (2016) has applied quantum uncertainty principle concepts in a proposed monetary theory. Khrennikov et al (2018) very recently has extended the concept of Heisenberg-Robertson inequality in the context of human decision making in general. The current paper builds upon the extant literature and extends to a central area of financial economics.

Ellsberg type uncertainty has not been studied carefully in the mainstream portfolio theory, and also the main stream theory uses standard probabilistic models to describe uncertainty. As has been explored in the review section these are the shortcomings of standard models. Hence the current paper is an early attempt to formulate a quantum like modeling of portfolio choice behavior under uncertainty.

Since the model proposed here is based on Ellsberg type uncertainties there is an empirical side to it, or either via experimental methods or via simulations we can obtain results of choice behavior of agents which can further be investigated from the model's predictions.

Overall, the points of departure from the standard decision-making theory are: introduction of inherent randomness in choice making, introduction of Non-Boolean logic in the form of Hilbert space formulation, scope of further experimental observations based on Ellsberg type uncertainties.

Objective or subjective probabilities?

There has been a profound debate at the heart of interpretation of QM, and much of it also revolves around the probabilistic picture of the world the frameworks depict. However over time generally two types of approaches have evolved: ontological and epistemological understandings of quantum state. Certainly there are deep issues, for example the difference between psi epistemic and psi ontic representations, where the former would mean there is presence of hidden variables (mostly local) which is incompletely described by the wave function or the quantum state, where as the ontic description would reject such sub quantum description of world. Then there are two specific Copenhagenish (to quote Leifer (2014)) schools, one relational quantum mechanics proposed first by Carlo Rovelli (1996 onwards) and two, QBism (Fuchs et al). Relational view holds that relation between states or systems in general is the only ontology of the world, where 'measurement' is nothing else but general interactions between 'observers' which are just passive systems getting correlated, probabilities are objective here. QBism on the other hand is a personalist and subjective 'user guide' or 'normative' decision theory, where QM is universal and can be used by any active agency (like a human rational decision maker) to update 'belief' states and assign coherently right probabilities to future events. Hence here probabilities are subjective.

Some authors have suggested that QBism might be more compatible with general quantum-like decision framework. Choice of QBism might be more natural since normativity is one hallmark of decision theory, for example in Arrow-Debrue asset pricing framework, the underlying assumption is that of revealed preference theory which is a normative theory based on optimization of utility functions. However the mathematical complexity would be to introduce SIC POVM (symmetric and informationally complete positive value measures) measures, which is central to the deductions of QBism. It's an open question to prove existence of such measures for any arbitrary dimension of Hilbert space.

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