A Farsighted Stable Set for Partition Function Games

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Abstract

In this paper, we introduce a concept of a farsighted stable set for a partition function game and interpret the union of all farsighted stable sets as the core of the game, to be called the strong-core, which reduces to the traditional core if the worth of every coalition is independent of the partition to which it belongs and the game is adequately represented by a characteristic function. We show that every farsighted stable set for a partition function game, like a characteristic function game, contains just a single feasible payoff vector, and the strong-core (i.e. the union of all farsighted stable sets) is nicely related to two previous core concepts for partition function games. Finally, we justify the farsighted stable sets also as a non-cooperative solution concept by showing that every farsighted stable set can be supported as an equilibrium outcome of an infinitely repeated game.

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1. Introduction

In the cooperative approach to game theory, the conventional game primitive is a characteristic function which, if utilities are transferable, assigns a real number to each coalition -- called the worth of the coalition. But a characteristic function cannot model situations in which the payoff of each coalition depends on other coalitions that may form in the complement. In fact, externalities from coalition formation are an important feature of many situations for which the cooperative approach to game theory otherwise appears appropriate. E.g., an important feature of treaties on climate change, such as the Kyoto Protocol, is that the signatories' payoffs depend not only on the actions taken by them but also on actions taken by the non-signatories. Similarly, benefits from mergers in oligopolistic markets depend on how the other outside firms react. The partition function (Thrall and Lucas, 1963) is a way of presenting information about these externalities. A partition function, if utility is transferable, also assigns a real number to each pair comprising a coalition and a partition to which the coalition belongs – called the worth of the coalition in the partition. Since the worth of each coalition in a game in characteristic function form is independent of what other coalitions form, they are special cases of games in partition function form.

In view of the generality and applications of the partition function games, a recently active literature is concerned with extensions of the solution concepts for characteristic function games to partition function games. But other than the extensions of the core and the Shapley value[1], no similar extension seems to have been proposed for stable sets -- introduced originally by von Neumann and Morgenstern (1944) as a solution for characteristic function games, and modified later by Harsanyi (1974) and Ray and Vohra (2014). That is perhaps because until recently there were few persuasive applications of the stables sets. But with the publication of Ray and Vohra (2014), interest in stable sets seems to have revived. They show that the farsighted stable sets, proposed as a modification of the stable sets in Harsanyi (1974), are applicable to the important class of simple games among others and closely related to the core of the game in that every farsighted stable set consists of a single core payoff vector and the union of all farsighted stable sets is "almost" equal to the core.

[1] See Chander and Tulkens (1997), Maskin (2003), and Hafalir (2007) for extensions of the core and Maskin (2003), de Clippel and Serrano (2008), and McQuillin (2009) among others for extensions of the Shapley value.

In this paper, we motivate and introduce farsighted stable sets for a *partition function* game and interpret the union of all farsighted stable sets as the core of the game, to be called the strong-core, which reduces to the traditional core if the worth of every coalition is independent of the partition to which it belongs and the game is adequately represented by a characteristic function. We show that every farsighted stable set of a partition function game, like that of a characteristic function game, is a singleton, and the strong-core (i.e. the union of all farsighted stable sets) is nicely related to two previous core concepts, namely, the γ- and the δ- cores of partition function games. More specifically, we show that the strong-core is a stronger concept than the γ-core, i.e., the strong-core $\subset$ γ-core in general and provide an example in which the inclusion is strict.

Since in most applications partition function games can be classified as games with either negative or positive externalities (see Yi, 1997, Maskin, 2003, and Hafalir, 2007 among others for this classification), we characterize the strong-core separately for each of these classes. For games with positive externalities, we show that the strong-core is a strictly weaker concept than the δ-core but a strictly stronger concept than the γ-core, i.e., δ-core $\subset$ strong-core $\subset$ γ-core and there are examples in which both inclusions are strict. For games with negative externalities, we show that the strong-core is a strictly stronger concept than the δ-core, but equal to the γ-core, i.e., γ-core $\subset$ strong-core $\subset$ δ-core and there are examples in which the second inclusion is strict. Since the strong-core $\subset$ γ-core in general, it follows that for games with negative externalities the first inclusion is not strict and the strong-core coincides with the γ-core. Thus, for partition function games which can be classified as games with positive or negative externalities, the strong-core sits strictly between the γ- and δ- cores except in the case of negative externalities when it is equivalent to the γ-core, but strictly smaller than the δ-core.

Since the sufficient conditions for the existence of a traditional core of a characteristic function game are known to be also sufficient for the existence of nonempty γ- and δ- cores, the above characterization of the strong-core implies that in games with positive or negative externalities the same conditions are also sufficient for the existence of a non-empty strong-core and, therefore, also for the existence of the farsighted stable sets. However, for completeness, we also derive sufficient conditions for the existence of a nonempty strong-core for games which do not exhibit positive or negative externalities. More specifically, we introduce a notion of partial superadditivity which is weaker than superadditivity and show that in partially superadditive partition function games, the strong-core is equal to the γ-core and, therefore, the well-known necessary and sufficient condition for a characteristic function game to admit a nonempty core is also necessary and sufficient for the existence of a nonempty strong-core and, thus, for the existence of a farsighted stable set. As will be shown, this sufficient condition is weaker than a previous sufficient condition for a partition function game to admit a nonempty γ-core (Hafalir, 2007: Proposition 2). In addition, we show that convexity of a partition function game is also sufficient for the existence of a nonempty strong-core and, thus, of a farsighted stable set.

A growing branch of the literature seeks to unify cooperative and non-cooperative approaches to game theory through underpinning cooperative game theoretic solutions with non-cooperative equilibria, the “Nash Program” for cooperative games.[2] In the same vein, we show that each farsighted stable set can be supported as an equilibrium outcome of a non-cooperative game. This game is intuitive and consists of infinitely repeated two-stages. In the first stage of the two-stages, which begins from the finest partition as the status quo, each player announces whether he wishes to stay alone or form a nontrivial coalition with the other players. In the second stage of the two-stages, the players form a partition as per their announcements. The two-stages are repeated if the outcome of the second stage is the finest partition from which the game began in the first place.

The paper is organized as follows. In Section 2, we introduce the notation and definition of farsighted stable sets for partition function games and interpret the union of the farsighted stable sets as the strong-core. We show that a prominent class of partition function games admit nonempty strong-cores and, thus, there exists a farsighted stable set for these games. In Section 3, we consider partition function games with negative or positive externalities and characterize the strong-core relative to the γ- and δ- cores. In section 4, we introduce the notion of a partial superadditive game and introduce two sufficient conditions for the existence of a nonempty strong-core and, therefore, a farsighted stable set. In Section 5, we introduce an infinitely

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[2] Analogous to the microfoundations of macroeconomics, which aim at bridging the gap between the two branches of economic theory, the Nash program seeks to unify the cooperative and non-cooperative approaches to game theory. Numerous papers have contributed to this program including Rubinstein (1982), Perry and Reny (1994), Pérez-Castrillo (1994), Compte and Jehiel (2010), and Lehrer and Scarsini (2013) among others.

repeated coalition formation game. We show that every farsighted stable set can be supported as an equilibrium outcome of the game. Section 6 draws the conclusion.

2. Farsighted stable sets for partition function games

Let $N = \{1, \dots, n\}, n \geq 3$, denote the set of players. A set $P = \{S_1, \dots, S_m\}$ is a partition of N if $S_i \cap S_j = \emptyset$ for all $i, j = 1, \dots, m, i \neq j$, and $\cup_{i=1}^{m} S_i = N$. We shall denote the finest partitions of $N, S,$ and $N \backslash S$ by $[N], [S]$, and $[N \backslash S]$, respectively, the cardinality of set S by $|S|$, and (to save on notation) the sets $\{i\}, \{S\}$,$\{N \backslash S\}$, and $\{N\}$ simply by $i, S, N \backslash S$, and N, respectively, whenever no confusion is possible.

A partition function is a real valued function of a coalition and a partition and denoted by $v(S; P)$ where P is a partition of N and S is a member of P. We shall denote a partition function game by a pair (N, v). Since the worth of a coalition in a partition function game depends on the partition to which the coalition belongs, the partition function games are sometimes referred to as games with externalities. A partition function game in which the worth of every coalition is independent of the partition and depends *only* on the coalition can be considered as a special case and adequately represented by a characteristic function, i.e., by a game which is "externalities free".

Given a partition function game (N, v), a *feasible payoff vector* is a vector $x = (x_1, \dots, x_n)$ such that $\sum_{i \in N} x_i = v(N; \{N\})$. In words, a feasible payoff vector represents a division of the worth of the grand coalition. Similarly, a vector $y = (y_1, \dots, y_n)$ is a feasible payoff vector for a partition $P = \{S_1, \dots, S_m\}$ if $\sum_{k \in S_i} y_k = v(S_i; P), i = 1, \dots, m$. Thus, a feasible payoff vector for a partition permits transfers among the members of each coalition in the partition, but not across the coalitions.[3] We assume throughout the paper that each coalition in a partition is free to decide its part of the feasible payoff vector. Thus, a feasible payoff vector for a partition respects both "feasibility" and "coalitional sovereignty": the two natural requirements that, as emphasized by Ray and Vohra (2014), must be satisfied by a farsighted stable set. To indicate the partition for

[3] This means that the payoff of a coalition in a feasible payoff vector for a partition is equal to its worth in the partition.

which a payoff vector is feasible, we shall henceforth denote a feasible payoff vector x by (x, N) and a payoff vector y which is feasible for partition P by (y, P).

A partition function game (N, v) is *grand-coalition superadditive* if the worth of the grand coalition is at least as large as the sum of the worths of coalitions in any partition, i.e., $v(N; \{N\}) \geq \sum_{S_i \in P} v(S_i; P)$ for every partition $P = \{S_1, \dots, S_m\}$. Thus, if a partition function game is grand-coalition superadditive, then formation of the grand coalition is optimal and $\sum_{i \in N} x_i \geq \sum_{i \in N} y_i$ for any feasible payoff vectors (x, N) and (y, P). This means that if (N, v) is grand-coalition superadditive, then in every partition P there is at least one (singleton or non-singleton) coalition S which is "worse-off", i.e. $v(S; P) \leq \sum_{i \in S} x_i$ for some coalition $S \in P$.

2.1 *Farsighted dominance*

The objective of this section is to define farsighted dominance by feasible payoff vectors, i.e., by payoff vectors which are feasible for the grand coalition. A central idea underlying this notion is that the players may form a new partition from an existing one and each coalition in the new partition may adjust accordingly its part of the feasible payoff vector. More specifically, coalitions in a partition may split or merge to form a new partition. Some of the coalitions involved in this process may be thought of as "perpetrators" in the formation of the new partition and others might be "residual" coalitions of players left behind by the perpetrators. Formally, let $P = \{S_1, \dots, S_m\}$ be an existing partition, then P' is a partition formed from P if there is a coalition T such that $P' = \{T, S_1 \backslash T, \dots, S_m \backslash T\}$ where coalition T is the perpetrator and coalitions $S_i \backslash T, i = 1, \dots, m$, are the residuals. We shall denote formation of a partition P' with a feasible payoff vector y' from an existing partition P with a feasible payoff vector y by

$$(y, P) \underset{T}{\rightarrow} (y', P'),$$

where T is the perpetrator and y' is the feasible payoff vector independently chosen by the coalitions in the new partition P'.[4] It is worth considering the two extreme cases: a player leaves

[4] Since the worth of a coalition in a partition function game depends on the partition, the worth of even those coalitions which are "untouched" by the move of T may change, necessitating adjustments even in their parts of the feasible payoff vector.

a coalition and forms a singleton coalition, i.e., $|T| = 1$ or all coalitions merge, i.e., $T = \cup_{i=1}^{m} S_i = N$ and, thus, each $S_i \backslash T = \emptyset, i = 1, \dots, m.$ [5]

In order to rule out indifference on the part of coalitions or their members, we henceforth adopt the convention that the players strictly prefer to be members of the grand coalition than to be members of a coalition in a partition other than the grand coalition even if their payoffs are the same, i.e., player i is "better-off" as a member of the grand coalition with feasible payoff vector x than as a member of a coalition in a partition $P \neq \{N\}$ with feasible payoff vector y if $x_i \geq y_i$, but if $P = N$, i.e., x and y are both feasible for the grand coalition then player i is better-off under x then under y only if $x_i > y_i$. Accordingly, we define farsighted domination of a feasible payoff vector for a partition other than the grand coalition separately from that of a feasible payoff vector for the grand coalition.

A feasible payoff vector (x, N) *farsightedly dominates* a feasible payoff vector (y, P), $P \neq \{N\}$, if there is a sequence of feasible payoff vectors $(y^0, P^0), (y^1, P^1), \dots, (y^q, P^q)$, where $(y^0, P^0) = (y, P)$ and $(y^q, P^q) = (x, N)$, and a corresponding sequence of coalitions T^h such that for each $h = 1, \dots, q$:

$$(y^{h-1}, P^{h-1}) \underset{T^h}{\rightarrow} (y^h, P^h)$$

and

$$x_i \geq y_i^{h-1} \text{ for each } i \in T^h.$$

In words, there could be several steps in moving from the feasible payoff vector $(y, P), P \neq N$, to the feasible payoff vector (x, N). Farsighted dominance requires that every member of each coalition that makes a move at some step must be better-off at the end of the process.[6] What matters to the members of coalitions involved in moving the process are their "final payoffs" – and not their payoffs at the intermediate stages. Farsighted dominance of a feasible payoff vector by another is defined similarly:

[5] We shall follow the convention that $\{N, \emptyset, \dots, \emptyset\} \equiv \{N\}$.

[6] Since by our convention, a player prefers to be a member of the grand coalition than of a coalition in a partition other than the grand coalition even if the payoffs are the same.

A feasible payoff vector (x, N) farsightedly dominates a feasible payoff vector (x', N) if there is a sequence of feasible payoff vectors $(y^0, P^0), (y^1, P^1), \dots, (y^q, P^q)$, with $(y^0, P^0) = (x', N)$ and $(y^q, P^q) = (x, N)$, and a corresponding sequence of coalitions T^h such that for each:

$$(y^{h-1}, P^{h-1}) \underset{T^h}{\rightarrow} (y^h, P^h),\ h = 1, \dots, q,$$

$$x_i > x'_i, i \in T^1, \text{ and } x_i \geq y_i^h \text{ for each } i \in T^h, h = 2, \dots, q.$$

The inequalities are strict for $h = 1$, since the members of the initial perpetrator T^1 are (to begin with) members of the grand coalition and, therefore, they will not defect from the grand coalition unless their final payoffs are strictly higher .

2.2 *Farsighted stable sets for partition function games*

The above two farsighted dominance relations lead to the following definition of a farsighted stable set for a partition function game analogous to a farsighted stable set for a characteristic function game.[7]

Definition 1 A set of feasible payoff vectors F is a farsighted stable set for a partition function game if it satisfies:

Internal Stability. No feasible payoff vector in F is farsightedly dominated by another feasible payoff vector in F.

External Stability. Every feasible payoff vector *not* in F is farsightedly dominated by some feasible payoff vector in F.

Béal et al. (2008) show that every Harsanyi stable set (Harsanyi, 1974) for a characteristic function game consists of a single imputation. Similarly, Ray and Vohra (2014) motivate and introduce a concept of a farsighted stable set for characteristic function games and show that every farsighted stable set consists of a single core payoff vector. Now we show that a farsighted stable set for a partition function game also consists of a single feasible payoff vector.

[7] Also see Chwe (1994) and Béal et al. (2008) among others.

Theorem 1 Given a partition function game (N, v), a singleton set containing a feasible payoff vector $(x, N)$is a farsighted stable set if (i) for every partition $P = \{S_1, \dots, S_m\} \neq [N]$, $\sum_{j\in S_i} x_j \geq v(S_i; P)$ for at least one non-singleton coalition $S_i \in P$ and (ii) for the finest partition $[N]$, $x_i \geq v(i; [N]), i = 1, \dots, n$.

<u>Proof</u>: Let (x, N) be a feasible payoff vector as hypothesized. We prove that $\{x\}$ satisfies both internal and external stability. A singleton set trivially satisfies internal stability as there are no two distinct feasible payoff vectors in the set and consequently there is no possibility of farsighted dominance of x by another feasible payoff vector in the set. We prove external stability in two parts: (a) each (x, N) as hypothesized farsightedly dominates every feasible payoff vector $(y, P), P \neq N$, and (b) each (x, N) farsightedly dominates every other feasible payoff vector (x', N).

(a) Let $P = \{S_1, \dots, S_m\} \neq N$ and $(y^0, P^0) \equiv (y, P)$. If $P = [N]$, then $(y^0, P^0) \underset{T^1}{\rightarrow} (y^1, P^1) \equiv (x, N)$, where $T^1 = \cup_{j=1}^m S_j = N$, and $y_i^1 = x_i \geq y_i^0 = y_i$ for each $i \in T^1 = N$, since $x_i \geq v(i, [N]) = y_i, i = 1, \dots, n$. Thus, (x, N) farsightedly dominates (y, P), if $P = [N]$. If $P \neq [N]$, then P^0 includes at least one non-singleton coalition such that at least one member of the coalition is worse-off, i.e., $y_i^0 \leq x_i$ for at least some i. Let $T^1 = \{i\}$, $P^1 = \{T^1, S_1 \backslash T^1, \dots, S_m \backslash T^1\}$, and y^1 be a feasible payoff vector for the partition P^1. If $P^1 = [N]$, then $(y^0, P^0) \underset{T^1}{\rightarrow} (y^1, P^1) \underset{T^2}{\rightarrow} (y^2, P^2) = (x, N)$, where $T^2 = T^1 \cup \cup_{j=1}^m S_j \backslash T^1 = N$, and $y_i^0 \leq x_i$ and $y_j^1 \leq x_j, j \in T^2 = N$, since $x_i \geq v(i, [N]) = y_i^1, i = 1, \dots, n$. Thus, (x, N) farsightedly dominates (y, P). If $P^1 \neq N$, then, proceeding similarly, there exists a sequence $(y^0, P^0) \underset{T^1}{\rightarrow} (y^1, P^1) \underset{T^2}{\rightarrow} (y^2, P^2) \underset{T^3}{\rightarrow} \dots \underset{T^{q-1}}{\longrightarrow} (y^{q-1}, P^{q-1}) \underset{T^q}{\rightarrow} (y^q, P^q) = (x, N)$, where $|T^h| = 1, h = 1, \dots, q-1, T^q = N$, and $y_j^h \leq x_j, j \in T^h, h = 1, \dots, q$. Thus, (x, N) farsightedly dominates (y, P).

(b) Since (x, N) and (x', N) are both feasible payoff vectors and $(x, N) \neq (x', N)$, we have $x_i' < x_i$ for at least some i. Let $T^1 = \{i\}$, $P^1 = \{i, N \backslash i\}$ and y^1 a feasible payoff vector for the partition P^1. Then, there exists a sequence $(y^1, P^1) \underset{T^2}{\rightarrow} (y^2, P^2) \underset{T^3}{\rightarrow} \dots \underset{T^{q-1}}{\longrightarrow} (y^{q-1}, P^{q-1}) \underset{T^q}{\rightarrow} (y^q, P^q) = (x, N)$, where $|T^h| = 1, h = 1, \dots, q-1, T^q = N$, and $y_j^h \leq x_j, j \in T^h, h = 1, \dots, q$. But this implies that there also exists a sequence $(x', N) = (y^0, P^0) \underset{T^1}{\rightarrow} (y^1, P^1)$

$\underset{T^2}{\rightarrow} (y^2, P^2) \underset{T^3}{\rightarrow} \dots \underset{T^{q-1}}{\longrightarrow} (y^{q-1}, P^{q-1}) \underset{T^q}{\rightarrow} (y^q, P^q) = (x, N)$, where $|T^h| = 1, h = 1, \dots, q-1, T^q = N$, $y_j^h \leq x_j, j \in T^h, h = 2, \dots, q$, $x_i' < x_i$ and $T^1 = \{i\}$. Thus (x, N) farsightedly dominates (x', N). ■

It is worth interpreting conditions (i) and (ii) of the theorem. First, condition (i) does not follow from grand-coalition superadditivity of the game in that it requires that the "worse-off" coalition in a partition must be a *non-singleton* coalition rather than just *any* coalition. Second, condition (ii) is analogous to individual rationality of imputations in a characteristic function game and it indeed reduces to that if the worth of every coalition is independent of the partition and the game is adequately represented by a characteristic function. It is also worth noting from the proof of the theorem that for any deviation from a feasible payoff vector (x, N) belonging to a farsighted stable set to a feasible payoff vector for a partition (y, P), i.e., $(x, N) \underset{S}{\rightarrow} (y, P)$ where $P = \{S, N \backslash S\}$, there exists a dominance chain $(y, P) = (y^0, P^0) \underset{T^1}{\rightarrow} (y^1, P^1) \underset{T^2}{\rightarrow} (y^2, P^2) \underset{T^3}{\rightarrow} \dots \underset{T^{q-1}}{\longrightarrow} (y^{q-1}, P^{q-1}) \underset{T^q}{\rightarrow} (y^q, P^q) = (x, N)$, where $|T^h| = 1, h = 1, \dots, q-1, T^q = N$, and $y_j^h \leq x_j, j \in T^h, h = 1, \dots, q$, i.e., there exists a dominance chain such that members of the initial deviating coalition are not better-off as there final payoffs are the same. In other words, every deviation from a feasible payoff vector (x, P) belonging to a farsighted stable set is farsightedly "deterred" by (x, P) itself.

We illustrate the concepts so far by showing that there exists a farsighted stable set in a well-known class of partition function games. These games are symmetric, grand-coalition superadditive, and such that larger coalitions in each partition have lower per-member payoffs (see e.g. Ray and Vohra, 1997, Yi, 1997, and Chander, 2007). We show that the feasible payoff vector with equal shares is a farsighted stable set.

Theorem 2 Let (N, v) be a symmetric partition function game such that for every partition $P = \{S_1, \dots, S_m\}$, $v(S_i; P)/|S_i| < (=) v(S_j; P)/|S_j|$ if $|S_i| > (=) |S_j|, i, j \in \{1, \dots, m\}$ and $v(N; \{N\}) > \sum_{S_i \in P} v(S_i; P)$. Then, the feasible payoff vector with equal shares is a farsighted stable set.

Proof: Let $(x_1, \dots, x_n)$ be the feasible payoff vector with equal shares, i.e., $\sum_{i \in N} x_i = v(N; N)$ and $x_i = x_j, i, j \in N$. We claim that $(x_1, \dots, x_n)$ is a farsighted stable set.

Let $P = \{S_1, \dots, S_m\} \neq N$ be some partition of N. If $P = [N]$, then $x_i \geq v(i; [N])$ for all $\{i\} \in [N]$, since $v(N; N) > \sum_{i \in N} v(i; [N])$, $\sum_{i \in N} x_i = v(N; N)$, $v(i; [N]) = v(j; [N])$, and $x_i = x_j$ for all $i, j \in N$. If $P \neq [N]$, then the number of coalitions in the partition is $m \geq 2, m < n$. Without loss of generality assume that $|S_1| \geq |S_2| \geq \cdots \geq |S_m|$. Thus, $n > m \geq 2$ and $\sum_{i=1}^{m} v(S_i; P) < v(N; N) = \sum_{i \in N} x_i$, as hypothesized. This inequality implies $v(S_1; P) < \sum_{i \in S_1} x_i$, since $v(S_1; P)/|S_1| \leq v(S_j; P)/|S_j|$ for all $S_j \in P$ and $x_i = x_j, i, j \in N$. Since $n \geq 3$ and $P \neq [N], N$, we must have $|S_1| \geq 2$. This proves that each partition $P \neq [N]$, includes at least one non-singleton coalition which is worse-off relative to the feasible payoff vector with equal shares $(x_1, \dots, x_n)$ and $x_i \geq v(i; [N]), i = 1, \dots, n$. By Theorem 1, $\{(x_1, \dots, x_n)\}$ is a farsighted stable set. ■

The proof of Theorem 1 brings forth a conceptual issue which is common to the concepts of farsighted stable sets for both partition and characteristic function games, since it implicitly assumes "optimistic behavior" on the part of deviating coalitions in the sense that every deviating coalition in a dominance chain is assumed to proceed with the deviation if its members are better-off in at least *one* of the ultimate outcomes, whereas a conservative coalition would not proceed with the deviation unless *every* possible ultimate outcome makes its members better-off.[8] However, in the present context this issue is restricted to dominance chains which begin from a feasible payoff vector for the grand coalition and terminate at another feasible payoff vector for the grand coalition. If the chain starts from a feasible payoff vector for a partition other than the grand coalition, then the members of every deviating coalition in the chain are better-off no matter at which farsighted stable set the dominance chain terminates. Thus the problem of having to choose between multiple dominance chains can be avoided by defining instead a concept of a farsighted conservative stable set as the union of all farsighted stable sets. It is easily verified that this set satisfies internal stability if the deviating coalitions are conservative as well as external stability, since it is the union of all farsighted stable sets. In other words, the union of

[8] The difficulty of dealing with multiple continuation paths following an initial move also crops up in Greenberg (1990) where he discusses "optimistic" and "conservative" notions of dominance.

all farsighted stable sets is a farsighted conservative stable set of a partition function game which does not assume optimistic behavior on the part of deviating coalitions.[9] But we will not pursue this interpretation here. Since the union of all farsighted stable sets is closely related to two previous core concepts for partition function games, we interpret it instead as a core concept, but do not ignore the fact that it is the union of all farsighted stable sets and, thus, a farsighted stable set exists if and only if the so-defined core is non-empty.

2.3 *Farsighted stable sets and the core*

The core, proposed by Gillies (1953) almost ten years after von Neumann and Morgenstern (1944) introduced the stable sets, is a leading and influential solution concept for characteristic function games. But in a partition function game, unlike a characteristic function game, a deviating coalition has to take into account what other coalitions may form in the complement subsequent to its deviation, since its payoff depends on the entire partition. Therefore, all existing core concepts for a partition function game without fail make one or the other *ad hoc* assumption concerning the coalitions that may form in the complement subsequent to a deviation -- leading to alternative core concepts depending on the assumption made in this regard. In this section, we first review the two most widely used core concepts and then interpret the union of all farsighted stable sets as the core, to be called the strong-core of a partition function game, which does not assume formation of any particular partition subsequent to a deviation.

Definition 2 The γ-core of a partition function game (N, v) is the set of all feasible payoff vectors $(x_1, \dots, x_n)$ such that in every partition $\{S, [N\backslash S]\}, S \subset N, \sum_{i \in S} x_i \geq v(S; \{S, [N\backslash S]\})$.

The γ-core (Chander and Tulkens, 1997) assumes formation of a specific partition subsequent to a deviation from the grand coalition. In particular, it assumes that if coalition S deviates from the grand coalition then the partition $\{S, [N\backslash S]\}$ forms, and a γ-core payoff vector

[9] Since, as noted in the discussion following the proof of Theorem 1, any deviation from a feasible payoff vector forming a farsighted stable set is "deterred" by the feasible payoff vector itself, the union of all farsighted stable sets can also be seen as a partition function game analog of Chwe's (1994) largest consistent set of a characteristic function game.

is such that the deviating coalition S is worse-off in this partition.[10] But why should the complement of a deviating coalition break apart into singletons? This assumption of the γ-core has been much commented upon and debated in the literature and for this reason alternative core concepts for partition function games have been proposed.[11]

Definition 3 The δ-core of a partition function game (N, v) is the set of feasible payoff vectors $(x_1, \dots, x_n)$ such that in every binary partition $\{S, N\backslash S\}$, $S \subset N$, $\sum_{i \in S} x_i \geq v(S; \{S, N\backslash S\})$.

The δ-core (Maskin, 2003), like the γ-core, also assumes formation of a specific partition subsequent to a deviation from the grand coalition. Specifically, it assumes that if coalition S deviates from the grand coalition then the binary partition $\{S, N\backslash S\}$ forms, and a δ-core payoff vector is such that the deviating coalition S is worse-off in this partition.[12]

All other existing core concepts for partition function games similarly make one or the other *ad hoc* assumption regarding the partition that may form subsequent to a deviation. [13] Apart from these concepts, the traditional α- and β- cores (Aumann, 1961) not only assume formation of the binary partition $\{S, N\backslash S\}$ subsequent to a deviation by coalition S, but also that the complementary coalition $N\backslash S$ takes actions that minmax or maxmin the payoff of the deviating coalition S without regard to its own payoff.[14]

Definition 4 The strong-core of a partition function game (N, v) is the union of all farsighted stable sets, i.e., the set of all feasible payoff vectors $(x_1, \dots, x_n)$ such that in every partition $P = \{S_1, \dots, S_m\} \neq [N]$, $\sum_{j \in S_i} x_j \geq v(S_i; P)$ for at least one non-singleton coalition $S_i \in P$ and for the finest partition $[N]$, $x_i \geq v(i; [N]), i = 1, \dots, n$.

[10] In keeping with our convention, a coalition in a partition other than the grand coalition is worse-off even if it has the same payoff as in the grand coalition.
[11] See e.g. Rajan ((1989) and Hafalir (2007) who labels the γ-core differently as the s-core.
[12] Actually, the definition implicitly requires that not only S but the complementary coalition $N\backslash S$ also must be worse-off, since a deviation by $N\backslash S$ from the grand coalition would result in the binary partition $\{N\backslash S, S\}$.
[13] See Hafalir (2007) for a comprehensive list of alternative core concepts for partition function games. Also, the definitions of γ- and δ- cores presently do not correspond to the definitions in Hart and Kurz (1983), as Hart and Kurz only consider what happens when one member deviates from one coalition.
[14] See Chander (2007) and Ray and Vohra (1997) for additional criticisms of α- and β- cores.

In words, a feasible payoff vector belongs to the strong-core if in every partition at least one non-singleton coalition is worse-off and in the finest partition all coalitions (singletons) are worse off. The strong-core can be justified on purely technical grounds as a concept which, unlike the previous core concepts, makes no ad hoc assumption regarding the partition that may form subsequent to a deviation. But it can also be motivated and interpreted independently of other concepts:

Suppose that a proposal $x = (x_1, \dots, x_n)$ is under discussion of the grand coalition and must be collectively accepted or rejected. Now suppose that a coalition S thinks that it can do better than $(x_1, \dots, x_n)$ provided that a particular partition forms and a payoff vector $(y_1, \dots, y_n)$ which is feasible for the partition is chosen. The question is: what are the *minimum* requirements that the alternative proposal y must fulfil for S to succeed in convincing all concerned about it.[15] Clearly, a *necessary* condition for the alternative proposal y to be acceptable to all concerned is that no non-singleton coalition T must be worse-off, i.e., there must be no non-singleton coalition T such that $\sum_{i \in T} y_i = v(T) \le \sum_{i \in T} x_i$.[16] To put it differently, a (singleton or non-singleton) coalition has an *objection* to a proposal if there is a partition in which it is better-off and no non-singleton coalition is worse-off.[17] A feasible payoff vector belongs to the strong-core, if no coalition has an objection.

Definition 5 The strong-core of a partition function game (N, v) is the set of all feasible pay-off vectors $(x_1, \dots, x_n)$ such that no coalition has an objection.

Definition 5 is technically equivalent to Definition 4, since it implies that *every* singleton coalition $\{i\}$ must be worse-off in the finest partition (otherwise, the singleton coalition will have an objection) and in all other partitions at least some non-singleton coalition must be worse-off (otherwise, a non-singleton coalition in a partition will have an objection). It is also easily seen that the strong-core is consistent with the traditional core in the sense that it reduces to the

[15] It must convince all concerned, since its worth/payoff depends on what everyone else does.
[16] In view of Theorem 1, this condition also means that the coalitions are farsighted and conservative in the sense that a coalition does not deviate from the grand coalition if its deviation could generate a dominance chain which terminates at a feasible payoff vector in which some member of the coalition is not better-off.
[17] Thus, on the one hand, an objection is easy to find since an objecting coalition is allowed to assume formation of any partition, but, on the other hand, an objection is difficult to find since the payoff of every non-singleton coalition in the partition must be higher.

traditional core if the worth of every coalition is independent of the partitions to which it belongs and the partition function is adequately represented by a characteristic function. The following example illustrates the strong-core.

Example 1 Let $N = \{1,2,3,4\}$ and $v(S; P) = |S|^3(|N| - |P|)$, $S \subset N$.

In this game, $v(N; N) = 192$, $v(i; [N]) = 0$, and the feasible payoff vector $(0,64,64,64)$ belongs to the strong-core. Coalition $\{1\}$ has no objection to this payoff vector, since a non-singleton coalition is worse-off in each partition $P \neq N$ of which $\{1\}$ is a member. In particular, $v(N\backslash 1; \{1, N\backslash 1\}) = 54 < 64 + 64 + 64$ (since $|N\backslash 1| = 3$ and $|P| = 2$) and for every two-player coalition $S \subset \{2,3,4\}$, we have $v(S; \{1, S, N\backslash S\backslash 1\}) = 8 < 64 + 64$ (since $|S| = 2$ and $|P| = 3$). Finally, coalition $\{1\}$ itself is worse-off in the finest partition, since its payoff is the same as in the grand coalition with feasible payoff vector $(0,64,64,64)$ and, by our convention, a coalition is worse off in a partition other than the grand coalition if its payoff is the same as in the grand coalition.

Since in every partition $\{S, [N\backslash S]\} \neq [N]$ at most coalition S is a non-singleton, a strong-core payoff vector is also a γ-core payoff vector. Thus, the strong-core of a partition function game is a stronger concept than the γ-core, i.e., strong-core $\subset$ γ-core in general. But the two are not equivalent as the following example shows.

Example 2 Let $N = \{1,2, \dots, 5\}, v(N; N) = 13$, $v(S; \{S, [N\backslash S]\}) = 2.4s$, $v(S; \{S; N\backslash S\}) = 2.6s$ for $s < 4$, $v(S; \{S; N\backslash S\}) = 2.4s$ for $s = 4$, for each partition $P = \{ij, kl, m\}$, $v(ij; P) = v(kl; P\} = 6$ and $v(m; P) = 1$, for each partition $P = \{i, j, k, lm\}$, $v(i; P) = 1$, and for each partition $P = \{i, j, klm\}, v(i; P) = 1$.

In this game, the feasible payoff vector $(x_1, x_2, \dots, x_5) = (2.6, 2.6, \dots, 2.6)$ belongs to the γ-core and thus the γ-core is nonempty. However, the strong-core is empty. This is seen as follows: A feasible payoff vector $(x_1, x_2, \dots, x_5)$, by definition, belongs to the strong-core only if $\sum_{i \in N} x_i = 13$, $x_i \geq 2.4$, $i = 1,2, \dots, 5$, and at least for the partition $P = \{12,34,5\}$, either $v(12; P) \leq x_1 + x_2$ or $v(34; P) \leq x_3 + x_4$. But there can be no such feasible vector, since $x_i \geq 2.4$, $i = 1,2, \dots, 5$ and, therefore, $x_1 + x_2 = 13 - x_3 - x_4 - x_5 \leq 5.8 < v(12; P)$ and

$x_3 + x_4 = 13 - x_1 - x_2 - x_5 \leq 5.8 < v(34; P)$. Hence, the strong-core is empty, but the γ-core is not. Thus, the strong-core is strictly smaller than the γ-core. This is because the γ-core is determined *only* by the payoffs $v(S; \{S, [N \setminus S]\}, S \subset N$, and, unlike the strong-core, independent of the payoffs $v(S; P)$, $P = \{ij, kl, m\}, S \in P$. Thus, externalities from coalition formation play a greater role in the determination of the strong-core.

However, in three-player partition function games, as can be easily checked, the strong core is generally equal to the γ-core. In four-player games also the strong-core is equal to the γ-core if the game is grand-coalition is superadditive. For this reason we chose an example with five players to demonstrate that the two are different. The strong core is also comparable to the δ-core if the externalities are positive or negative.

3. Games with negative or positive externalities

In most applications, the partition function games can be divided into two separate categories (see e.g. Yi, 1997, Maskin, 2003, and Hafalir, 2007):

A partition function game (N, v) has *negative* (resp. *positive*) *externalities* if for every $P = \{S_1, \dots, S_m\}$ and $S_i, S_j \in P$, we have $v(S_k; P \setminus \{S_i, S_j\} \cup \{S_i \cup S_j\}) \leq (\text{resp.} \geq)\ v(S_k; P)$ for each $S_k \in P, k \neq i, j$.

In words, a partition function game has negative (resp. positive) externalities if a merger between two coalitions in a partition decreases (resp. increases) the worths of other coalitions in the partition.[18]

Theorem 3 (a) For partition function games with positive externalities, δ-core $\subset$ strong-core $\subset$ γ-core, and (b) for games with negative externalities, γ-core $\subset$ strong-core $\subset$ δ-core.

Proof: (a) First, suppose contrary to the assertion that in a game with positive externalities a δ-core payoff vector $(x_1, \dots, x_n)$ does not belong to the strong-core. Since $(x_1, \dots, x_n)$ belongs to the δ-core, for every coalition $S \subset N$ and partition $\{S, N \setminus S\}$, $\sum_{i \in S} x_i \geq v(S; \{S, N \setminus S\}) \geq v(S; \{S, [N \setminus S])$, since externalities are positive. In particular, for $S = \{i\}$, $x_i \geq v(i; [N]), i =$

[18] It is easily verified that the externalities in the game in Example 2 are not negative.

$1, \dots, n$. Furthermore, since $(x_1, \dots, x_n)$, by supposition, does not belong to the strong-core, there must exist a partition $P = \{S_1, \dots, S_m\} \neq [N]$ such that $v(S_i; P) > \sum_{j \in S_i} x_j$ for all $S_i \in P$ with $s_i > 1$. Then, since externalities are positive, $v(S_i; P') > \sum_{j \in S_i} x_j$, where $P' = \{S_i, N \backslash S_i\}$. But this contradicts that $(x_1, \dots, x_n)$ belongs to the δ-core. Hence our supposition is wrong and, therefore, every δ-core payoff vector $(x_1, \dots, x_n)$ belongs to the strong-core. This proves δ-core $\subset$ strong-core.

Second, if $(x_1, \dots, x_n)$ belongs to the strong-core, then, by definition, $\sum_{i \in S} x_i \geq v(S; [N \backslash S])$ for all non-singleton coalitions S and for the partition $P = [N]$, $x_i \geq v(i; [N]), i = 1, \dots, n$. Thus, every strong-core payoff vector $(x_1, \dots, x_n)$ belongs to the γ-core. This proves strong-core $\subset$ γ-core.

(b) First, let $(x_1, \dots, x_n)$ be a γ-core payoff vector of a partition function game (N, v) with negative externalities. We claim that $(x_1, \dots, x_n)$ also belongs to the strong-core. Suppose not. Since $(x_1, \dots, x_n)$ belongs to the γ-core, for every partition $\{S, [N \backslash S]\}, S \subset N$, $\sum_{i \in S} x_i \geq v(S; \{S, [N \backslash S]\})$ and $x_i \geq v(i; [N]), i = 1, \dots, n$. Then, since $(x_1, \dots, x_n)$, by supposition, does not belong to the strong-core, there must be a partition $P = \{S_1, \dots, S_m\} \neq [N]$ such that $v(S_i; P) > \sum_{j \in S_i} x_j$ for all $S_i \in P$ with $s_i > 1$. Let $P' = \{S_i, [N \backslash S_i]\}$ denote the partition in which all but coalition S_i is a singleton. Then, since the game (N, v) has negative externalities, $v(S_i; \{S_i, [N \backslash S_i]\}) \geq v(S_i; P) > \sum_{j \in S_i} x_j$. But this contradicts that $(x_1, \dots, x_n)$ is a γ-core payoff vector. Hence, our supposition is wrong and each γ-core payoff vector $(x_1, \dots, x_n)$ also belongs to the strong-core. This proves γ-core $\subset$ strong-core.

Second, suppose contrary to the assertion that for a game with negative externalities a strong payoff vector $(x_1, \dots, x_n)$ does not belong to the δ-core. Then, we must have $\sum_{i \in S} x_i < v(S; \{S, N \backslash S\})$ for some $S \subset N$. Since externalities are negative, this implies $\sum_{i \in S} x_i < v(S; \{S, [N \backslash S]\})$ for some $S \subset N$. But this contradicts that $(x_1, \dots, x_n)$ belongs to the strong-core. Thus our supposition is wrong and every strong-core payoff vector $(x_1, \dots, x_n)$ also belongs to the δ-core. This proves strong-core $\subset$ δ-core. ■

The theorem implies that if externalities are negative, the strong-core is a stronger concept than the δ-core but weaker than the γ-core, and if externalities are positive then the strong-core is a stronger concept than the γ-core but weaker than the δ-core. The theorem also implies that if externalities are negative, the strong-core is equal to the γ-core. This is seen as follows.

Corollary 1 For partition function games with negative externalities, the strong-core is equal to the γ-core.

Proof: For games with negative externalities, Theorem 3 implies that the γ-core is a subset of the strong-core. Since the strong-core, by definition, is a subset of the γ-core in general, it follows that the two are equal if externalities are negative. ■

We show by means of examples that the inclusion relationships δ-core ⊂ strong-core and strong-core ⊂ γ-core in part (a) of Theorem 3 are strict and in part (b) the inclusion relationship: strong-core ⊂ δ-core is also strict.

Example 3 Let $N = \{1,2,3\}, v(N; N) = 15,\ v(i; \{i, jk\}) = 1, v(jk; \{i, jk\}) = 9, v(1; \{1, [N\backslash 1]\}) = v(2; \{2, [N\backslash 2]\}) = 2$, and $v(3; \{3, [N\backslash 3]\}) = 9$.

The game in this example has negative externalities. The strong-core of this game is empty. But the δ-core is nonempty, since the feasible payoff vector (5, 5, 5) belongs to the δ-core. It follows that for games with negative externalities, the inclusion relationship: strong-core ⊂ δ-core in part (b) of Theorem 3 is strict.

Example 4 Let $N = \{1,2,3\}$, $v(N; \{N\}) = 24$, $v(i; [N]) = 1, i = 1,2,3$, $v(i; \{i, jk\} = 9$ for $\{i, j, k\} = N$, $v(12; \{12,3\}) = 12, v(13; \{13,2\}) = 13$, and $v(23; \{23,1\}) = 14$. [19]

The game in this example has positive externalities. The strong-core is nonempty, since the feasible payoff vector (7.5, 8, 8.5) belongs to the strong-core. But the δ-core is empty and, therefore, for games with positive externalities, the inclusion relationship: δ-core ⊂ strong-core in part (a) of Theorem 3 is strict.

[19] This example is a minor variation of an example previously considered in Maskin (2003) and de Clippel and Serrano (2008).

Example 5 Let $N = \{1,2,...,5\}$. $v(N;N) = 15$, $v(S;\{S,[N\backslash S]\} = 2.4s$, $v(S;\{S;N\backslash S\}) = 2.9s$ for each partition $= \{ij,kl,m\}$, $v(ij;P) = v(kl;P\} = 5.7$, $v(m;P) = 2.9$, for each partition $P = \{i,j,k,lm\}$, $v(i;P) = 2.9$, and for each partition $P = \{i,j,klm\}, v(i;P) = 2.9$.

This game has positive externalities. The feasible payoff vector (2.5, 2.5, 2.5, 2.5, **5**) belongs to the γ-core, but not to the strong-core -- confirming that the inclusion strong-core $\subset$ γ-core is strict. Furthermore, the feasible payoff vector (2.8, 2.8, 2.8, 2.8, **3.8**) belongs to the strong-core, but not to the δ-core – confirming that for games with positive externalities, the inclusion relations δ-core $\subset$ strong-core and strong-core $\subset$ γ-core can both be strict at the same time.

In summary, the strong-core is in general a stronger concept then the γ-core. For games with positive or negative externalities it sits between the γ- and the δ- cores and equal to the γ-core if externalities are negative. All inclusion relationships in Theorem 3, except one, are strict.

4. Existence of a farsighted stable set and a non-empty core

The purpose of this section is to exploit the inclusion relationships between the strong-core and the γ- and δ-cores to propose sufficient conditions for the existence of a nonempty strong-core and, thus, for the existence of a farsighted stable set, since every strong-core payoff vector, by definition, is a farsighted stable set.

Let $w^{\gamma}(S) = v(S;[N\backslash S])$, $S \subset N$. Then, w^{γ} is a restriction of the partition function v and the γ-core of (N,v) is equal to the core of the induced characteristic function game (N,w^{γ}). Similarly, the δ-core of (N,v) is equal to the core of the induced characteristic function game (N,w^{δ}), where $w^{\delta}(S) = v(S;\{S,N\backslash S\})$, $S \subset N$. This means that the γ-core (resp. the δ-core) is nonempty if and only if the induced characteristic function w^{γ} (resp. w^{δ}) is balanced (Bondareva, 1963 and Shapley, 1967).[20] Though the strong core is not similarly equal to the core of an induced characteristic function game, its relationship with the γ- and δ- cores, as established in Theorem 3, leads to two immediate sufficient conditions for the existence of a nonempty strong-core and thereby a farsighted stable set.

[20] This result is known as the the Bondareva-Shapley theorem. See Helm (2001) for an elegant application of this theorem.

Corollary 2 A partition function game (N, v) admits a non-empty strong-core if it has negative (resp. positive) externalities and the induced characteristic function game (N, w^{γ}) (resp. (N, w^{δ})) is balanced.

Proof: Since Corollary 1 shows that the strong-core of a partition function game (N, v) with negative externalities is equal to the γ-core, which in turn, as noted above, is equal to the core of the induced characteristic function game (N, w^{γ}), it follows from the Bondareva-Shapley theorem that the strong-core is nonempty if the induced characteristic function game (N, w^{γ}) is balanced.

Since Theorem 3 shows that the δ-core of a partition function game (N, v) with positive externalities is a subset of the strong-core, the strong-core is nonempty if the δ-core is. Since, as noted above, the δ-core is equal to the core of the induced characteristic function game (N, w^{δ}), it follows from the Bondareva-Shapley theorem that the strong-core of a partition function game with positive externalities is nonempty if the core of the induced characteristic function game (N, w^{δ}) is balanced and, therefore, the δ-core is nonempty. ■

Corollary 3 A partition function game (N, v) with negative (resp. positive) externalities admits a nonempty strong-core only if the induced characteristic function game (N, w^{δ}) (resp. (N, w^{γ})) is balanced.

Proof: If a partition function game has negative externalities, then, by Theorem 3, strong-core $\subset$ δ-core. Therefore, the game admits a nonempty strong-core only if the δ-core is nonempty, i.e. the induced characteristic function game $\left(N, w^{\delta}\right)$ is balanced. Similarly, if a partition function game has positive externalities, then, by Theorem 3, strong-core $\subset$ γ-core and therefore it admits a nonempty strong-core only if the γ-core is nonempty, i.e., the induced characteristic function game (N, w^{γ}) is balanced. ■

Though, as noted above, partition function games in most applications can be classified as games with either negative or positive externalities and the existence of a nonempty strong-core can be established by using the sufficient conditions in Corollary 2, it might still be useful to propose sufficient conditions that can be applied independently of the nature of externalities.

Now we propose two such sufficient conditions. We need the following concept which is weaker than the familiar concept of a superadditive partition function.[21]

Definition 6 A partition function game (N, v) is partially superadditive if for each partition $P = \{S_1, \dots, S_m\}$ with $|S_i| \geq 2, i = 1, \dots, k,$ and $|S_j| = 1, j = k+1, \dots, m, k \leq m,$ $\sum_{i=1}^{k} v(S_i; P) \leq v(S; P')$ where $P' = P \backslash \{S_1, \dots, S_k\} \cup \{\cup_{i=1}^{k} S_i\}$.

Partial superadditivity, as the term suggests, is weaker than the familiar notion of superadditivity which requires that combining *any arbitrary* coalitions increases their worth. In contrast, partial superadditivity requires that combining only *all non-singleton* coalitions increases their worth. Clearly, partial superadditivity is weaker than superaddivity. It is trivially satisfied by all partition function games with three players and also by four-player grand-coalition superadditive games. As in the case of characteristic function games (see e.g. Friedman, 1990), partial superadditivity is neither necessary nor sufficient for a partition function game to admit a nonempty strong-core : it is not necessary follows from the fact that the partition function games in Theorem 3 are not partially superadditive, but, as shown, admit a nonempty strong-core, and it is not sufficient follows from the fact that every three-player partition function game is partially superadditive, but not every three-player game admits a nonempty strong-core as Example 3 above shows.

Theorem 4 Let (N, v) be a partially superadditive partition function game. Then the strong-core is equal to the γ-core.

Proof: The strong-core, by definition, is a subset of the γ-core. More specifically, if $(x_1, \dots, x_n)$ belongs to the strong-core, then, by definition, $\sum_{i \in S} x_i \geq v(S; [N \backslash S])$ for all non-singleton coalitions S and $x_i \geq v(i; [N]), i = 1,2, \dots, n$. Therefore, $(x_1, \dots, x_n)$ also belongs to the γ-core. Thus, we only need to prove that each γ-core payoff vector also belongs to the strong-core.

Let $(x_1, \dots, x_n)$ be a γ-core payoff vector and let $P = \{S_1, \dots, S_m\}$ be a partition of N. If $P = \{S_1, \dots, S_m\} \neq [N]$, then let $|S_i| > 1$, for $i = 1, \dots, k$ and $|S_j| = 1$ for $j = k+1, \dots, m, k \leq$

[21]See de Clippel and Serrano (2008) for a formal definition of a superadditive partition function. Hafalir (2007) uses the term "fully cohesive" in place of "superadditive".

m, and $S = \cup_{i=1}^{k} S_i$. Since v is partially superadditive, $\sum_{i=1}^{k} v(S_i; P) \leq v(S; P')$ where $P' = P \backslash \{S_1, \dots, S_k\} \cup \{S\}$. Clearly, $P' = \{S, [N \backslash S]\}$. Since $(x_1, \dots, x_n)$ is a γ-core payoff vector, $\sum_{i \in S} x_i \geq v(S; \{S, [N \backslash S]\}) = v(S; P') \geq \sum_{i=1}^{k} v(S_i; P)$. This inequality can be rewritten as $\sum_{i=1}^{k} \sum_{j \in S_i} x_j \geq \sum_{i=1}^{k} v(S_i; P)$ and, therefore, $\sum_{j \in S_i} x_j \geq v(S_i; P)$ for at least one $S_i \in \{S_1, \dots, S_k\} \subset P$ with $s_i > 1$. If $P = [N]$, then since $(x_1, \dots, x_n)$ is a γ-core payoff vector, $x_i \geq v(i; \{i, [N \backslash i]\}) = v(i; [N])$. This proves that $(x_1, \dots, x_n)$ belongs to the strong-core. ■

Corollary 4 The strong-core is equal to the γ-core, if the game has three-players or the game has four players and grand-coalition superadditive.

Proof: Partition function games with three players, by definition, are partially superadditive. Also a four-player game is partially superadditive, if it is grand-coalition superadditive. Hence, Theorem 4 implies the corollary. ■

Corollary 5 A partition function game (N, v) admits a non-empty strong-core if it is partially superadditive and the induced characteristic function game (N, w^{γ}) is balanced.

Proof: Since Theorem 4 shows that the strong-core of a partially superadditive partition function game is equal to the γ-core which in turn, as noted above, is equal to the core of the induced characteristic function game (N, w^{γ}), the strong-core is nonempty if the core of the characteristic function game (N, w^{γ}) is nonempty, i.e., the induced characteristic function game (N, w^{γ}) is balanced (by the Bondareva-Shapley theorem). ■

Propositions 1 and 2 in Hafalir (2007) show that a convex partition function is superadditive and admits a nonempty γ-core. Since a superadditive partition function, by definition, is partially superadditive and the γ-core is nonempty only if the induced characteristic function game (N, w^{γ}) is balanced, by the Bondareva-Shapley theorem, it follows that the sufficient conditions in Corollary 5 are weaker than convexity of the partition function assumed in Hafalir (2007) for proving the existence of a nonempty γ-core.

Convexity of a characteristic function game is known to be a sufficient condition for the game to admit a nonempty core (Shapley, 1971). We show that convexity of a partition function game is similarly sufficient for existence of a nonempty strong-core.

Corollary 6 A convex partition function game (N, v) admits a nonempty strong-core.

Proof: First, if the partition function is convex, then it is superadditive (Hafalir, 2007: Proposition 1) and, therefore, the partition function is partially superadditive. Second, if the partition function game is convex, the induced characteristic function game (N, w^{γ}) is convex (Hafalir, 2007: Proposition 2) and, therefore, it admits a nonempty core (Shapley, 1971) and, therefore, balanced, by the Bondareva-Shapley theorem. The proof now follows from Corollary 5, since a convex partition function game (N, v) is partially superadditive and the induced characteristic function game (N, w^{γ}) is balanced. ■

Since the *strong-core* is a subset of the γ-core in general, Corollary 6 is a stronger result then Proposition 2 in Hafalir (2007) which shows that the γ-*core* is nonempty if the partition function is convex. Since, as noted, the sufficient conditions in Corollary 5 are weaker than convexity of the partition function, Corollary 5 proves a stronger result under weaker sufficient conditions than those in Proposition 2 in Hafalir (2007).

Theorem 3 and its corollaries imply that in partially superadditive partition function games a significant amount of information is strategically redundant. This is especially true in the case of three-player partition function games, since they are always partially superadditive, and also in the case of four-player grand-coalition superadditive games. However, for games with five or more players, as Example 2 illustrates, the same information may not be redundant.

5. Non-cooperative foundations of farsighted stable sets

In this section, we assume that the members of a coalition in a partition may choose to not give effect to their coalition and in that case their payoffs are the same as in a partition in which each member of the coalition is a singleton and all other coalitions in the partition are the same. The motivation for this assumption comes from the fact that for partition function games which are derived from a strategic game (Ichiishi, 1981, Ray and Vohra, 1997), not giving effect to a coalition in a partition is equivalent to the members of the coalition to deliberately choose the same strategies that they would if they were all singletons, given the strategies of the other coalitions in the partition.[22]

We also assume, without loss of generality, that $v(S; P) > 0$ for all partitions P and all coalitions $S \in P$. Though our analysis is independent of how a coalition in a partition divides its worth, but to be concrete we assume that each coalition in a partition divides its worth proportionally to a feasible payoff vector $(x_1^*, \dots, x_n^*)$ in the sense that for each partition $P = \{S_1, \dots, S_m\}$ and each coalition $S_i \in P$, the payoff of each player j in coalition S_i is $x_j \equiv x_j^* \times [v(S_i; P)/\sum_{k \in S_i} x_k^*]$. It will be made clear below that the analysis does not depend on this assumption and it holds for any arbitrary division of each coalition's worth.

4.1 An *infinitely repeated game*

We show that every farsighted stable set can be supported as an equilibrium outcome of a non-cooperative game. This game, to be called the *infinitely repeated game* or simply the *repeated game*, consists of infinitely repeated two-stages. The first stage of the two-stages begins from the finest partition $[N]$ as the status quo and each player announces either 0 or some positive integer from 1 to n. In the second stage of the two-stages, all those players who announced the same positive integer in the first stage form a coalition.[23] All those players who announced 0 remain singletons.[24] If the outcome of the second stage is not the finest partition, the game ends and the partition formed remains formed forever.[25] But if the outcome of the second stage is the finest partition -- as in the status quo from which the game began in the first

[22] See Chander (2007) for an actual example of a partition function derived from a strategic game and what not giving effect to a coalition in a partition means in terms of actions/strategies of the members of the coalition.
[23] Thus the players can form any partition by announcing appropriate numbers; and no partition, other than the finest, can be formed without the consent of all players, since any player can unilaterally effect a change in the partition to be formed by suitably changing its announcement. In contrast, formation of the finest partition with the announcement of 0 by *every* player cannot be *unilaterally* changed by a player, since a non-singleton coalition can be formed only if at least *two* players announce the same *positive* integer.
[24] Thus a player can choose to stay alone by simply announcing 0 and cannot be forced to form a coalition with any other player or players.
[25] This is analogous to the rule in the infinite bargaining game of alternating offers (Rubinstein, 1982) in which the game ends if the players agree to a split of the pie, but continues, possibly *ad infinitum*, if they disagree. It is also the same as the rule that formation of a non-trivial partition is irreversible (e.g. Compte and Jehiel, 2010).

place -- the two-stages are repeated, possibly *ad infinitum*, until some partition other than the finest is formed in a future round.[26] In either case, the players receive payoffs in each period proportionally to a feasible payoff vector $(x_1^*, \ldots, x_n^*)$.

It may be noted from the description above that the repeated game allows the players to form *any* partition other than the finest and end the game; it does not rule out *a priori* any partition as a possible equilibrium outcome. The trivial/finest partition $[N]$ can be an outcome of the second stage of the two-stages if i) all players announce zero in the first stage of the two-stages, ii) no two players announce the same positive integers in the first stage of the two-stages and iii) two or more players announce the same positive integers, but decide in the second stage of the two-stages to not give effect to their coalitions. Since, as noted, a partition other than the finest can be formed only with the consent of all players, formation of a partition other than the finest is to be interpreted as an agreement among all players. In contrast, formation of the finest partition is to be interpreted as a disagreement.

To describe the repeated game in more concrete terms, visualize the following scenario: All players meet in a negotiating room to decide on the formation of a partition knowing in advance what their payoffs will be in each partition. They may form a partition other than the finest or they may all decide to stay alone, i.e., form the trivial partition. If the players agree to form a non-trivial partition, the meeting ends, the players receive per-period payoffs according to a pre-specified rule, and all leave the room. But if the players do not agree to form a non-trivial partition, the meeting and negotiations continue and nobody leaves the room until the players agree to form a non-trivial partition.

We assume that the payoffs are discounted and the discount factor $\delta < 1$ is sufficiently large. Since the structure of the continuation game is exactly the same as the original game, we restrict ourselves to equilibria in stationary strategies of the repeated game. In fact, since the game ends as soon as a non-trivial partition is formed, only equilibria in stationary strategies are relevant. Accordingly, we characterize the equilibria of the repeated game by comparing only the per-period payoffs of the players. We do this first for a farsighted stable set consisting of a feasible

[26] Since the game starts from the finest partition, not allowing repetition of the two-stages if the outcome of the second stage is again the finest partition would be inconsistent.

payoff vector $(x_1^*, \ldots, x_n^*)$ is *interior* in the sense that $x_i^* > v(i; [N]), i = 1, \ldots, n$, and for each partition $P = \{S_1, \ldots, S_m\} \neq [N], \{N\}$, $\sum_{j \in S_i} x_j^* > v(S_i; P)$ for at least some $S_i \in P$ with $|S_i| \geq 2$, [27] and then note that the same result also holds for feasible payoff vectors which are not interior, if as per our convention the players strictly prefer to be members of the grand coalition than of a coalition in a partition even if their payoffs are the same.

Theorem 5 Given a partition function game (N, v), a farsighted stable set consisting of an interior feasible payoff vector $(x_1^*, \ldots, x_n^*)$ is an equilibrium outcome of the repeated game if the worth of each coalition is divided proportionally to $(x_1^*, \ldots, x_n^*)$ and the discount factor δ is sufficiently close to 1.

Proof: Since $v(S; P) > 0$ for every partition P and all coalitions $S \in P$, we have $x_i^* > 0, i = 1, \ldots, n$. We show that in the repeated game,

(i) not to give effect to a coalition if it does not include all players is an equilibrium strategy of every member of the coalition, and

(ii) the grand coalition is the unique equilibrium outcome and the players' per-period equilibrium payoffs are equal to $(x_1^*, \ldots, x_n^*)$.

It is convenient to prove the theorem separately for $n = 3$ and $n > 3$.

Case $n = 3$: We show that (i) implies (ii) and then prove that the strategies in (i) are indeed equilibrium strategies, since they imply (ii). Given the strategies in (i) and players' responses to them, we derive a reduced form of the infinitely repeated game as follows:

Given the strategies in (i), let $(w_1, \ldots, w_n)$ denote the players' undiscounted per-period stationary strategies equilibrium payoffs in the repeated game. (a) If in some period, all players do not announce the same positive integer or some player announces $i = 0$, then, as the strategies in (i) require, no non-singleton coalition is given effect by the players and the outcome is the finest partition implying undiscounted per period payoffs of $(w_1, \ldots, w_n)$, since the continuation game is identical to the original game. (b) If in some period, all players announce the same

[27] It may be noted that the feasible payoff vector with equal shares in the games in Theorem 2 is interior.

positive integer, then the outcome is the grand coalition, the game ends, and the players' undiscounted per-period payoffs are (x_1^*, x_2^*, x_3^*).[28]

In three-player games, there is no loss of generality if each player chooses only between strategies $i = 1$ and $i = 0$, since even with these restricted strategy sets the players can form all possible partitions by announcing either 1 or 0. Then, given the strategies in (i), the payoff matrix of the repeated game in reduced form is:

		Player 3 $i = 1$		**Player 3** $i = 0$	
		Player 2 $i = 1$	**Player 2** $i = 0$	**Player 2** $i = 1$	**Player 2** $i = 0$
Player 1	$i = 1$	x_1^*, x_2^*, x_3^*	$\delta w_1, \delta w_2, \delta w_3$	$\delta w_1, \delta w_2, \delta w_3$	$\delta w_1, \delta w_2, \delta w_3$
	$i = 0$	$\delta w_1, \delta w_2, \delta w_3$	$\delta w_1, \delta w_2, \delta w_3$	$\delta w_1, \delta w_2, \delta w_3$	$\delta w_1, \delta w_2, \delta w_3$

A solution to this reduced game can be found by considering a mixed strategy Nash equilibrium. Let p_1, p_2, p_3 be the probabilities assigned by the three players to the strategy $i = 1$. Then, in equilibrium each player, say 1, should be indifferent between strategies $i = 0$ and $i = 1$. Therefore, a mixed strategy equilibrium must be such that $w_1 = p_2 p_3 \delta w_1 + (1 - p_2 p_3)\delta w_1 = p_2 p_3 x_1^* + (1 - p_2 p_3)\delta w_1$. If $x_1^* > \delta w_1$, then the pure strategy $i = 1$ is the unique dominant strategy and the resulting payoff is $w_1 = x_1^*$, consistent with the inequality $x_1^* > \delta w_1$. Thus the pure strategy $i = 1$ for each player is the unique dominant equilibrium strategy, i.e., the reduced

[28] It may be noted that if the grand coalition is indeed an equilibrium outcome of the repeated game, then it will occur without delay. That is because the per-period payoffs of the players would be otherwise lower in the periods preceding the period in which the grand coalition is formed, since $x_i^* > v(i; [N])$.

game admits a unique dominant strategies equilibrium in pure strategies, the grand coalition is the unique equilibrium outcome, and the players' undiscounted per-period equilibrium payoffs are (x_1^*, x_2^*, x_3^*).

Now we prove that the strategies in (i) are indeed equilibrium strategies, since they imply (ii). Suppose in some period, two players, say 2 and 3, announce $i = 1$, but player 1 announces $i = 0$. Suppose further that in Stage 2, players 2 and 3, contrary to the strategies in (i), give effect to their coalition. Such a deviation from the strategies in (i) would lead to payoffs of $\left(\frac{x_2^*}{x_2^*+x_3^*}\right) v(23;\{23,1\}) < x_2^*$ and $\left(\frac{x_3^*}{x_2^*+x_3^*}\right) v(23;\{23,1\}) < x_3^*$ for players 2 and 3 (resp.), since the payoffs are proportional to $(x_1^*, \dots, x_n^*)$ and $x_2^* + x_3^* > v(23;\{23,1\})$ as (x_1^*, x_2^*, x_3^*) forms a farsighted stable set and is in the interior.[29] However, if players 2 and 3 adhere to the strategies in (i) and thus do not give effect to their coalition, then the game will be repeated and their payoffs, as shown, will be δx_2^* and δx_3^*, which for δ sufficiently close to 1 are higher than what their payoffs would be if they give effect to their coalition and thereby end the game. Thus, it is ex post optimal for both players 2 and 3 to not give effect to their coalition, which player 1 *must* take into account when deciding its strategy.[30]This proves (i) as well.

<u>Case $n > 3$</u>: The proof for (i) implies (ii) is identical to that for $n = 3$. Thus, we only need to prove that (ii) implies (i). Suppose contrary to the assertion that some players form a non-singleton coalition or coalitions other than the grand coalition and give effect to them. Let $P = \{S_1, \dots, S_m\} \neq [N], \{N\}$ be the resulting partition such that $|S_1| > 1$ and $|S_j| = 1, j = 2, \dots, n$. Then, since the vector $(x_1^*, \dots, x_n^*)$ forms a farsighted stable set and in the interior, $v(S_1; P) < \sum_{j \in S_1} x_j^*$ and the payoff of each $j \in S_1 \in P$ is equal to $\frac{x_j^*}{\sum_{i \in S_1} x_i^*} v(S_1; P) < x_j^*$. But if the members of S_1 were to not give effect to their coalition, then the resulting partition would be the finest, the game will be repeated and result in payoffs equal to δx_j^* for each $j \in S_1$ which for δ sufficiently close to 1 are higher . Therefore, dissolving S_1 is an equilibrium strategy for each member of S_1. Next, let P be such that $|S_1||S_2| > 1$ and $|S_j| = 1, j = 3, \dots, n$. Then, since

[29] To minimize notation, we denote coalitions $\{i, j\}$ and $\{k\}$ simply by ij and k, respectively.

[30] The argument here is not that players 2 and 3 can force player 1 to merge with them by threatening to not give effect to their coalition (and thus deny him the opportunity to free ride), but rather that given their strategies in (i) and the players' responses to it, such an action is ex post optimal for players 2 and 3, i.e., a subgame-perfect equilibrium strategy.

$(x_1^*, \dots, x_n^*)$ forms a farsighted stable set and in the interior, either $v(S_1; P) < \sum_{j \in S_1} x_j^*$ or $v(S_2; P) < \sum_{j \in S_2} x_j^*$ or both. Without loss of generality, let $v(S_2; P) < \sum_{j \in S_2} x_j^*$. Then the payoff of each $j \in S_2 \in P$ is $\frac{x_j^*}{\sum_{i \in S_2} x_i^*} v(S_2; P) < x_j^*$. If the members of coalition S_2 were to not give effect to their coalition, then S_1 will be the only non-singleton coalition in the resulting partition and, as shown, to not give effect to S_1 is an equilibrium strategy for each of its members. Thus, if the members of S_2 do not give effect to their coalition then the members of S_1 will also not give effect to their coalition resulting in the finest partition, repetition of the game, and the equilibrium payoff of δx_j^* for each $j \in S_2 \in P$ which is higher than $\frac{x_j^*}{\sum_{i \in S_2} x_i^*} v(S_2; P)$ if δ is sufficiently close to 1. Similarly, let $P = \{S_1, \dots, S_m\}$ be such that $|S_1|, \dots, |S_k| > 1$ and $|S_j| = 1, j = k + 1, \dots, n$. Then, the members of S_k will not give effect to their coalition, members of S_{k-1} will not give effect to their coalition, and so on … resulting in the finest partition, repetition of the game and equilibrium payoffs which are higher for every member of the non-singleton coalitions if δ sufficiently close to 1. This proves that (ii) implies (i) as it is ex post optimal for every member of *each* non-singleton coalition in every partition $P \neq [N], \{N\}$ to not give effect to their coalition. ■

It is easily checked that the theorem also holds if the payoffs are not discounted (i.e. $\delta = 1$), but beside the grand coalition, the finest partition is also an equilibrium outcome. The same is also true if the farsighted stable set consists of a feasible payoff vector which is not interior but as per our convention the players strictly prefer to be members of the grand coalition than of a coalition in a partition even if their payoffs are the same. However, this additional equilibrium is Pareto dominated and, thus, the grand coalition is the unique equilibrium outcome in these cases too if we assume that the players never play a Pareto dominated equilibrium. The theorem also holds for any arbitrary division of the worth of each coalitions in a partition, since at least one member in at least one non-singleton coalition in every non-trivial partition other than the grand coalition is worse-off. Such a player can be eventually better-off if it leaves the coalition and forms a singleton as then there will be another member of another coalition who would be similarly worse-off and can be eventually better-off if it leaves the coalition and this process, as in a dominance chain in Section 2.1, will continue until all non-singleton coalitions disintegrate and the finest partition is formed and the game is repeated.

The theorem implies that the grand coalition is the unique equilibrium outcome of the repeated game. In contrast, Ray and Vohra (1997) and Yi (1997) show that if the game is not repeated, then the grand coalition is not an equilibrium outcome. The intuition for their contrasting result is as follows: If the two-stages in, say, a three-player game are to be played only once, then for a singleton coalition i considering a unilateral deviation from the grand coalition, the strategically relevant coalition structure is $\{i, jk\}$, and not the finest partition $\{i, j, k\}$, since the strategies of the other two players j and k will not aim at the finest partition if the two-stages are not to be repeated and their payoffs in the partition $\{i, jk\}$ are higher than in the finest partition $\{i, j, k\}$. Therefore, if the payoff of a singleton coalition i in the partition $\{i, jk\}$ is higher than in the farsighted stable set, then i would gain by leaving the grand coalition as that would result in formation of the partition $\{i, jk\}$ and not the finest partition $\{i, j, k\}$.Thus, the three coalition structures with a pair and a singleton and not the grand coalition would be the equilibrium outcomes if the game is limited to a single play of the two-stages.

6. Conclusion

We have motivated and introduced two related concepts for partition function games, namely: the farsighted stable sets and the strong-core. The proposed farsighted stable sets respect both "feasibility" and "coalitional sovereignty" and the strong-core is nicely related to the previous core concepts for partition function games, but unlike them does not arbitrarily assume formation of a specific partition subsequent to a deviation from the grand coalition. Thus, the strong-core seems to settle a long-standing debate on which core concept to use in applications of partition function games. The two concepts are closely related in that every farsighted stable set consists of a single strong-core payoff vector and each strong-core payoff vector forms a farsighted stable set. This suggests that the strong-core has powerful farsighted stability property.

There are intriguing similarities, as well as some contrasts, with the farsighted stable sets and the core for characteristic function games in that the farsighted stable sets for a partition function game, much like the farsighted stable sets for characteristic function games in Ray and Vohra

(2014), are singletons, respect feasibility and coalitional sovereignty, and subsets of the core. The strong-core reduces to the traditional core if the worth of every coalition is independent of the partition to which it belongs and the partition function is adequately represented by a characteristic function. In contrast, the Harsanyi stable sets (Hatsanyi, 1974) for characteristic function games, though also singletons, do not respect coalitional sovereignty and are disjoint from the core (see Ray and Vohra, 2014, p.2-3).

We showed that the strong core is generally stronger than the γ-core and also comparable to the δ-core if a game has positive or negative externalities. More specifically, we showed that for partition function games with positive externalities, δ-core $\subset$ strong-core $\subset$ γ-core and for games with negative externalities, γ-core $\subset$ strong-core $\subset$ δ-core, and these inclusions, except one, are strict. Similarly, the strong-core is equal to the γ-core if the game is partially superadditive. On the one hand, these results lead to sufficient conditions for the existence of a nonempty strong core as well as a farsighted stable set and, on the other hand, they imply that significant amounts of information in partition function games with three or four players as well as those with negative externalities may be redundant. But as Example 2 shows that is not true in games with five or more players and positive externalities.

The paper also contributes to the “Nash program” for cooperative games in that it was shown that the farsighted stable sets and the strong-core payoff vectors can be supported as equilibrium outcomes of an intuitive infinitely repeated game. The fact that the game is repeated infinitely many times plays a crucial role for the result to hold. Finally, a forthcoming paper proves that an oligopoly with more than four firms admits a nonempty strong-core which is a subset of the γ-core, since the corresponding partition function, though grand-coalition superadditive, is neither partially superadditive nor exhibits negative externalities.[31]

[31] Rajan (1989) proves that if the number of firms in an oligopoly is less than or equal to four, then the γ-core, and thus the strong core, is nonempty.

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