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Global solutions to the discrete nonlinear breakage equations without mass transfer

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A R T I C L E I N F O

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A B S T R A C T

Global existence of mild solutions to the discrete collisional breakage equations is established for a broad class of collision kernels, without imposing any growth assumptions. In addition, classical solutions are constructed, and uniqueness is proved for an appropriate class of kinetic coefficients and initial data. The large time behavior of solutions is also discussed.

1. Introduction

Coagulation-fragmentation processes naturally occur in the dynamics of cluster growth and describe the way a system of clusters can merge to form larger ones or fragment to form smaller ones. Models of cluster growth arise in a wide variety of situations, including aerosol science, astrophysics, colloidal chemistry, polymer science, and biology. In the model considered in this paper, clusters are assumed to be identified by a single parameter, their size, which ranges in the set of positive integers $\mathbb{N}\setminus\{0\}$. Equivalently, each cluster is made of a finite number of identical elementary units and this number is usually referred to as their size. In the following, we shall refer to i-clusters for clusters made of i elementary units, $i\geq 1$. In contrast, the size of clusters may take any value in $(0,\infty)$ in the so-called continuous model.

On the one hand, coagulation is inherently nonlinear, as two or more clusters merge to form a larger cluster (see Fig. 1.1). On the other hand, fragmentation or breakage can be classified into two categories: linear (spontaneous) fragmentation and nonlinear (collision-induced) fragmentation. In the former process, a cluster breaks apart, either spontaneously due to intrinsic instabilities, or through external perturbations,

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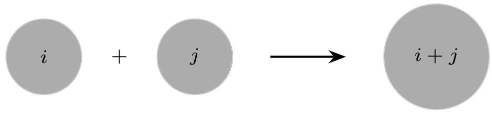

Fig. 1.1. Illustration of the coagulation process where a i-cluster and a j-cluster combine to form a $i+j$-cluster.

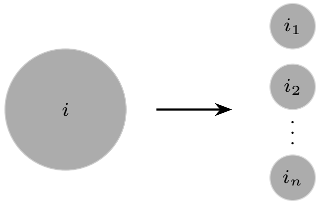

Fig. 1.2. Illustration of the fragmentation process without loss of matter, where a i-cluster breaks into smaller clusters with respective sizes $i_1, i_2, \ldots, i_n$, with the sum of their sizes being equal to that of the original particle.

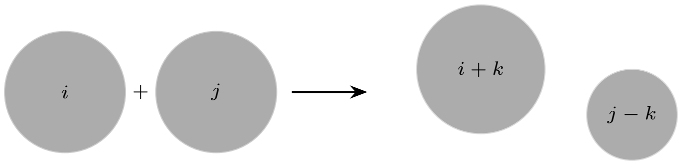

Fig. 1.3. Illustration of non-linear fragmentation process with mass transfer. During the collision, a k-cluster, $k < j$, is transferred from the j-cluster to the i-cluster, resulting in clusters of sizes $i+k$ and $j-k$.

such as mechanical stress or radiation (see Fig. 1.2). In nonlinear or collision-induced fragmentation, the collision of two clusters may lead to an exchange of mass between the clusters besides their splitting. A typical example of a collision-induced fragmentation event with mass transfer is the formation of clusters with respective sizes $i+k$ and $j-k$ after the collision of two clusters with respective sizes i and $j > k$, see Fig. 1.3, a more complicated example of fragmentation with mass transfer being depicted in Fig. 1.4. As a consequence, the maximal size of the clusters may increase when mass transfer is possible. Such a phenomenon does not take place in nonlinear fragmentation without mass transfer: the simplest situation in that case corresponds to the breakage of only one of the incoming clusters into smaller fragments while the other remains intact, see Fig. 1.5. However, in general, both clusters split into smaller fragments as depicted in Fig. 1.6.

A widely used approach in the modeling of these processes is based on rate equations, which track the time evolution of cluster size distributions. The first equation of this kind, modeling the coagulation phenomenon, was introduced by the Polish physicist M. Smoluchowski in his seminal papers [24,25]. The coagulation equation, both with and without linear fragmentation, has been extensively studied over the past few decades, the size variable being either discrete or continuous; for a detailed and thorough review, see [4] and the references therein.

In [20], Laurençot and Wrzosek study the discrete coagulation equation with nonlinear breakage, marking it as the first mathematical study addressing nonlinear breakage. More precisely, denoting by $\psi_i(t)$, $i \geq 1$, the number density of i-clusters at time $t \geq 0$, the discrete coagulation equation with nonlinear breakage reads

$$\frac{d\psi_i}{dt} = \frac{1}{2}\sum_{j=1}^{i-1} p_{j,i-j}\Gamma_{j,i-j}\psi_j\psi_{i-j} - \sum_{j=1}^{\infty}\Gamma_{i,j}\psi_i\psi_j + \frac{1}{2}\sum_{j=i+1}^{\infty}\sum_{k=1}^{j-1}(1-p_{j-k,k})\Phi^i_{j-k,k}\Gamma_{j-k,k}\psi_{j-k}\psi_k, \qquad i \geq 1, \tag{1.1a}$$

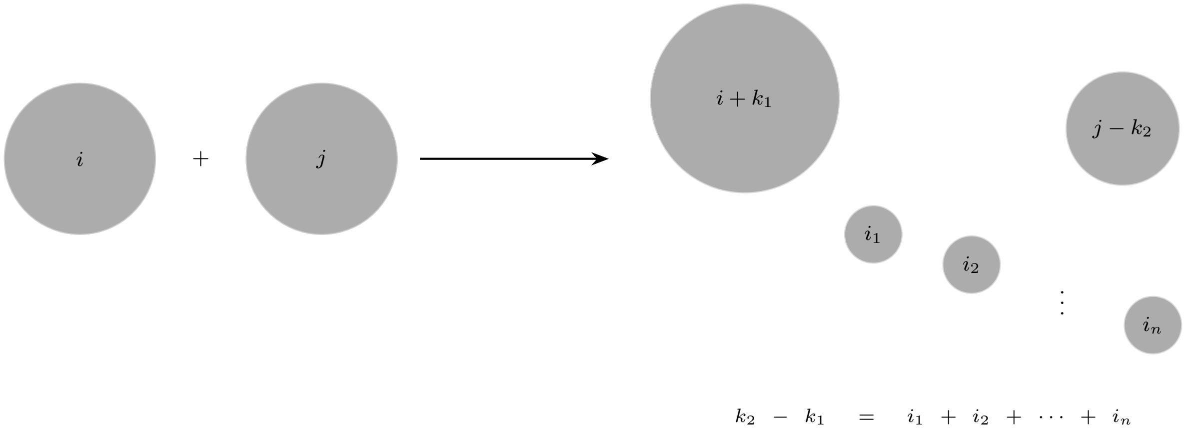

Fig. 1.4. Illustration of the nonlinear fragmentation process with mass transfer. Upon collision, the interaction results in net mass transfer from the j-cluster to the i-cluster, illustrated by size changes to $i+k_1$ and $j-k_2$, $k_1 \leq k_2 < j$, combined with the production of other fragments with respective masses $i_1, \ldots, i_n$, which sums up to the net transferred amount $k_2 - k_1$. No matter is lost overall.

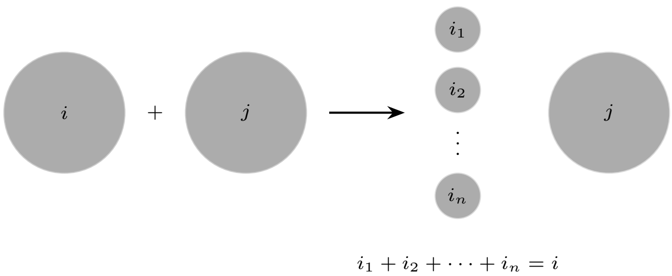

Fig. 1.5. Illustration of nonlinear fragmentation process without mass transfer and without loss of matter. During the collision, a i-cluster splits into smaller clusters of sizes $i_1, i_2, \ldots, i_n$ such that $i_1 + i_2 + \cdots + i_n = i$, while the j-cluster remains unchanged.

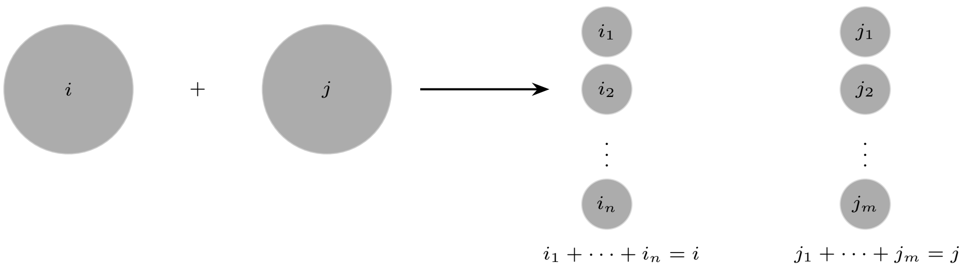

Fig. 1.6. Illustration of the nonlinear fragmentation process in a binary collision. Upon collision between clusters of sizes i and j, one or both may fragment into smaller clusters (shown here symmetrically for generality), with mass conserved separately for each: $i_1 + \cdots + i_n = i$ and $j_1 + \cdots + j_m = j$. The actual outcome depends on the collision kernel and fragment distribution function used in the model.

$$\psi_i(0) = \psi_i^{\text{in}}, \qquad i \geq 1. \tag{1.1b}$$

Here, $\Gamma_{i,j}$ denotes the rate of collisions between i-clusters and j-clusters, while $p_{i,j}$ represents the probability that two colliding clusters with respective sizes i and j merge into a single $i+j$-cluster. The complementary probability, $1 - p_{i,j}$, corresponds to cluster fragmentation, possibly involving a transfer of matter. The coefficients $(\Gamma_{i,j})$ and $(p_{i,j})$ satisfy the following symmetry property

$$0 \le p_{i,j} = p_{j,i} \le 1, \qquad \Gamma_{i,j} = \Gamma_{j,i} \ge 0, \qquad i,j \ge 1,$$

while $\{\Phi^s_{i,j}, s = 1, 2, ..., i+j-1\}$ is the size distribution function of the fragments resulting from the collision between a i-cluster and a j-cluster and satisfies

$$\begin{aligned} &\Phi^s_{i,j} = \Phi^s_{j,i} \ge 0, \quad 1 \le s \le i+j-1, \\ &\sum_{s=1}^{i+j-1} s\Phi^s_{i,j} = i+j, \end{aligned} \qquad i,j \ge 1. \tag{1.2}$$

The second identity in (1.2) ensures mass conservation during each collisional breakage event, so that conservation of matter is expected throughout time evolution. In terms of the number densities $(\psi_i)_{i\ge 1}$, mass conservation reads

$$\sum_{i=1}^{\infty} i\psi_i(t) = \sum_{i=1}^{\infty} i\psi_i^{\mathrm{in}}, \qquad t \ge 0. \tag{1.3}$$

The first term in (1.1a) accounts for the formation of i-clusters through coagulation, with a rate determined by the effective coagulation kernel $(p_{i,j}\Gamma_{i,j})$, whereas the second term represents the depletion of i-mers due to collisions with other clusters in the system. Finally, the third term describes the creation of i-clusters resulting from the collision and subsequent breakup of larger clusters. It is worth noting that the assumption (1.2) allows for mass transfer between the colliding clusters, meaning that some of the resulting clusters may be larger than either of the incoming clusters. In other words, mass transfer between the colliding clusters may occur and the mean size of the system of clusters does not necessarily decrease during the time evolution. Let us also mention here that, when $p_{i,j} = 1$, the equation (1.1) reduces to the classical Smoluchowski coagulation equation. In [20], the authors investigate the existence, uniqueness, mass conservation, and long time behavior of weak solutions to (1.1) under reasonable assumptions on the collision kernel and the daughter distribution function. The study performed in [20] also explores the occurrence of the gelation phenomenon; that is, the breakdown of the identity (1.3) in finite time. In [3], the previous work is extended to investigate classical solutions and explore various additional properties. The continuous counterpart of (1.1) has undergone significant study in recent years; see [6,7,11].

In the absence of coagulation ($p_{i,j} = 0$), the equation (1.1) becomes the discrete nonlinear fragmentation equation and reads

$$\frac{d\psi_i}{dt} = \frac{1}{2}\sum_{j=i+1}^{\infty}\sum_{k=1}^{j-1} \Phi^i_{j-k,k}\Gamma_{j-k,k}\psi_{j-k}\psi_k - \sum_{j=1}^{\infty}\Gamma_{i,j}\psi_i\psi_j, \qquad i \ge 1, \tag{1.4a}$$

$$\psi_i(0) = \psi_i^{\mathrm{in}}, \qquad i \ge 1. \tag{1.4b}$$

In [2], the well-posedness of (1.4) is studied for a broad class of collision kernels and daughter distribution functions which does not exclude mass transfer. In addition, non-trivial stationary solutions are constructed, a result which is closely related to mass exchange during collisions. We also mention here the analysis performed on the exchange-driven model in [10,22,23] and its generalized version in [5], where the outcome of the collision of a i-cluster and a j-cluster is, either a $i+k$-cluster and a $j-k$-cluster, or a $i-k$-cluster and a $j+k$-cluster for some fixed integer $k \ge 1$, the classical exchange-driven model corresponding to $k = 1$. Besides, the continuous version of (1.4) is investigated in [14].

A condition on the daughter distribution function $(\Phi^s_{i,j})$ excluding mass transfer is provided in [8] and reads

$$\Phi^s_{i,j} = \mathbf{1}_{[s,\infty)}(i)\varphi_{s,i;j} + \mathbf{1}_{[s,\infty)}(j)\varphi_{s,j;i}$$

for $i,j\ge 1$ and $1\le s\le i+j-1$, where $\mathbf{1}_{[s,\infty)}$ denotes the characteristic function of the interval $[s,\infty)$. In that case, the system (1.4) reduces to

$$\frac{d\psi_i}{dt} = \sum_{j=i+1}^{\infty}\sum_{k=1}^{\infty}\varphi_{i,j;k}\Gamma_{j,k}\psi_j\psi_k - (1-\delta_{i,1})\sum_{j=1}^{\infty}\Gamma_{i,j}\psi_i\psi_j, \qquad i\ge 1, \tag{1.5a}$$

$$\psi_i(0) = \psi_i^{\rm in}, \qquad i\ge 1, \tag{1.5b}$$

where $\delta_{1,1}=1$, $\delta_{i,1}=0$, $i\ge 2$, and $\{\varphi_{i,j;k}, 1\le i\le j-1\}$ denotes the distribution function of the fragments resulting from the collision of a j-cluster with a k-cluster and satisfies the conservation of matter

$$\sum_{i=1}^{j-1} i\varphi_{i,j;k} = j, \qquad j\ge 2, \quad k\ge 1. \tag{1.6}$$

Since each cluster fragments into smaller clusters after a collision, see Fig. 1.5, it is expected that, in the long time, only 1-clusters will remain.

Considerable attention has been given to the study of the continuous version of (1.5), focusing on its analytical solutions. In [8], Cheng and Redner investigate the asymptotic behavior of various classes of models, demonstrating that certain models can be transformed into the linear fragmentation equation through a suitable rescaling of time. This transformation is extensively applied in [9] to examine the nonlinear fragmentation equation with product collision kernels, addressing the existence and non-existence of solutions, as well as the emergence of finite time singularities. Further insights into the dynamics of the models analyzed in [8] are provided by Krapivsky and Ben-Naim in [17]. Additionally, the nonlinear fragmentation equation with both product and sum collision kernels is studied by Kostoglou and Karabelas in [15], employing a combination of analytical solutions and asymptotic expansions; see also [16]. From a mathematical perspective, the continuous version has been examined in [12,13], with a focus on the analysis of existence, non-existence, and the occurrence of the shattering phenomenon.

As for the discrete setting, the existence of classical solutions to (1.5) is established in [1] for collision kernels having at most quadratic growth

$$\Gamma_{i,j}\le Aij, \qquad i,j\ge 1, \tag{1.7}$$

and a broad class of fragment distribution functions. Various other properties, including uniqueness, differentiability, moment propagation, and large time behavior, are also investigated. Motivated by the recent study [19], where the Redner–ben-Avraham–Kahng (RBK) cluster system, also known as the cluster eating system, is investigated without imposing growth conditions on the kinetic coefficients, the aim of this work is to show that the growth condition (1.7) can be relaxed and that global mild solutions to (1.5) can be constructed assuming only the collision kernel $(\Gamma_{i,j})$ to be non-negative and symmetric; that is,

$$0\le \Gamma_{i,j}=\Gamma_{j,i}, \qquad i,j\ge 1. \tag{1.8}$$

However, the dissipation properties of the collision-induced fragmentation system are much weaker than those of the RBK cluster system and this sole assumption does not seem to be sufficient to derive an existence result for (1.5). Thus, as in [1], it requires to be supplemented with an additional assumption on the fragment distribution function, besides (1.6): there exist non-negative constants α_0 and α_1 such that

$$\varphi_{i,j;k}\le \alpha_0+\alpha_1\varphi_{i,k;j} \quad \text{for all } 1\le i\le j-1,\ k\ge j. \tag{1.9}$$

A possible physical meaning of (1.9) is in some sense the following: when a j-cluster and a k-cluster collide, see Fig. 1.5, the proportion of i-clusters, $1 \le i < \min\{j,k\}$, produced by the splitting of the $\min\{j,k\}$-cluster is smaller than that produced by the splitting of the $\max\{j,k\}$-cluster, up to the scaling factor α_1. Equivalently, when two clusters collide, the splitting of the largest one produces proportionally more small fragments that the splitting of the smaller one.

Remark 1.1. It is important to note that the assumption (1.9) holds for a wide range of fragment distribution functions. Indeed, typical examples include bounded fragment distribution functions such as

$$\varphi_{i,j;k} = \frac{i^\nu j}{\sum_{l=1}^{j-1} l^{1+\nu}}, \qquad 1 \le i \le j-1, \quad j \ge 2, \quad k \ge 1, \tag{1.10}$$

for $\nu \ge -1$, which reduces to the uniform distribution

$$\varphi_{i,j;k} = \frac{2}{j-1}, \qquad 1 \le i \le j-1, \quad j \ge 2, \quad k \ge 1,$$

for $\nu = 0$. It also includes unbounded fragment distribution functions like

$$\varphi_{i,j;k} = j\delta_{i,1}, \qquad 1 \le i \le j-1, \quad j \ge 2, \quad k \ge 1,$$

which satisfies (1.9) with $(\alpha_0, \alpha_1) = (0,1)$, the fragment distribution function (1.10) for $\nu < -1$, which satisfies (1.9) with

$$\alpha_0 = 0, \quad \alpha_1 = \begin{cases} 2^{2+\nu}, & \nu \in [-2,-1), \\ \sum_{l=1}^{\infty} l^{1+\nu}, & \nu < -2, \end{cases}$$

and

$$\varphi_{i,j;k} = \frac{1}{2^i} \frac{j 2^{j-1}}{2^j - j - 1}, \qquad 1 \le i \le j-1, \quad j \ge 2, \quad k \ge 1, \tag{1.11}$$

which satisfies (1.9) with $\alpha_0 = \alpha_1 = 1$, see Lemma A.1 in the Appendix.

As we shall see below, assumptions (1.6), (1.8), and (1.9) allow us to construct mild solutions to (1.5) for a large class of initial conditions, including all non-negative sequences having a finite superlinear moment. To be more precise, for $\sigma \ge 0$, let us define the Banach space

$$Y_\sigma := \left\{ \psi = (\psi_i)_{i \ge 1} : \qquad \psi_i \in \mathbb{R}, \ \sum_{i=1}^{\infty} i^\sigma |\psi_i| < \infty \right\}$$

with the norm

$$\|\psi\|_\sigma := \sum_{i=1}^{\infty} i^\sigma |\psi_i|, \qquad \psi \in Y_\sigma,$$

along with its positive cone

$$Y_{\sigma,+} := \{ \psi \in Y_\sigma : \psi_i \ge 0 \text{ for each } i \ge 1 \}.$$

In particular, for a non-negative cluster distribution ψ, the norm $\|\psi\|_0$ represents the total number of clusters, while the norm $\|\psi\|_1$ accounts for the total mass of the clusters, so that the conservation of mass (1.3) is equivalent to $\|\psi(t)\|_1 = \|\psi^{\text{in}}\|_1$ for all $t \geq 0$. We next introduce the set $\mathcal{G}_1$ of non-negative and convex functions $G \in C^2([0,\infty))$ such that $G(0) = G'(0) = 0$ and G' is a concave function, as well as

$$\mathcal{G}_{1,\infty} := \left\{ G \in \mathcal{G}_1 \ : \ \lim_{\zeta \to \infty} \frac{\zeta G'(\zeta) - G(\zeta)}{\zeta} = \infty \right\}.$$

It readily follows from the definition of $\mathcal{G}_{1,\infty}$ and l'Hospital rule that any $G \in \mathcal{G}_{1,\infty}$ satisfies

$$\lim_{\zeta \to \infty} G'(\zeta) = \lim_{\zeta \to \infty} \frac{G(\zeta)}{\zeta} = \infty. \tag{1.12}$$

Remark 1.2. One can check that the functions $z \mapsto z^m$ with $m \in (1,2]$ and $z \mapsto z\left[\ln\left(e^{m-1} + z\right)\right]^m$ with $m > 1$ belong to $\mathcal{G}_{1,\infty}$, but not $z \mapsto z\ln(1+z)$ (though it belongs to $\mathcal{G}_1$ and satisfies (1.12)).

With this notation, we may summarize the main contribution of this work as follows: given a collision kernel satisfying the symmetry and non-negativity condition (1.8) and a fragment distribution function satisfying (1.6) and (1.9), we construct a global mild solution to (1.5) for any initial condition $\psi^{\text{in}} \in Y_{1,+}$ such that

$$\sum_{i=1}^{\infty} G_0(i)\psi_i^{\text{in}} < \infty$$

for some function $G_0 \in \mathcal{G}_{1,\infty}$. In particular, the price to pay for having no growth condition on the collision kernel is that we cannot handle arbitrary initial data in $Y_{1,+}$. Still, the existence result applies to any initial condition ψ^{in} which belongs to $Y_{\sigma,+}$ for some $\sigma > 1$.

Before presenting the existence result, we first recall the definition of a global mild solution to (1.5) in $Y_{1,+}$.

Definition 1.3. Consider $\psi^{\text{in}} \in Y_{1,+}$. A global mild solution $\psi = (\psi_i)_{i\geq 1}$ to (1.5) is a sequence of non-negative functions in $L^\infty((0,\infty), Y_{1,+})$ satisfying, for each $i \geq 1$ and $t > 0$,

(i) $\psi_i \in C([0,\infty))$;

(ii)

$$\sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j\psi_k \in L^1((0,t)), \qquad \sum_{j=1}^{\infty} \Gamma_{i,j}\psi_i\psi_j \in L^1((0,t));$$

(iii)

$$\begin{aligned} \psi_i(t) = \psi_i^{\text{in}} &+ \int_0^t \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s)ds \\ &- (1 - \delta_{i,1}) \int_0^t \sum_{j=1}^{\infty} \Gamma_{i,j}\psi_i(s)\psi_j(s)ds. \end{aligned} \tag{1.13}$$

Theorem 1.4. *Assume that the kinetic coefficients $(\Gamma_{i,j})$ and $(\varphi_{i,j;k})$ satisfy the assumptions (1.8), (1.6), and (1.9). Consider $\psi^{\text{in}} \in Y_{1,+}$ such that*

$$\mathcal{J}_0 := \sum_{i=1}^{\infty} G_0(i)\psi_i^{\mathrm{in}} < \infty \tag{1.14}$$

for some $G_0 \in \mathcal{G}_{1,\infty}$. *Then there exists at least one global mild solution* ψ *to* (1.5) *in the sense of Definition 1.3 which additionally satisfies the mass conservation* (1.3)

$$\|\psi(t)\|_1 = \|\psi^{\mathrm{in}}\|_1, \qquad t \ge 0.$$

Moreover, for any non-negative sequence $(\Lambda_i)_{i\ge1}$ *such that the sequence* $(\Lambda_i/i)_{i\ge1}$ *is non-decreasing and*

$$M_\Lambda(\psi^{\mathrm{in}}) := \sum_{i=1}^{\infty} \Lambda_i\psi_i^{\mathrm{in}} < \infty, \tag{1.15}$$

then the mild solution $\psi = (\psi_i)_{i\ge1}$ *constructed above satisfies the uniform-in-time tail estimate*

$$\sum_{i=r}^{\infty} \Lambda_i\psi_i(t) \le \sum_{i=r}^{\infty} \Lambda_i\psi_i^{\mathrm{in}} \quad \text{for all } t > 0 \ \text{ and } \ r \ge 1. \tag{1.16}$$

A straightforward consequence of Theorem 1.4 is that, given $\sigma > 1$ and $\psi^{\mathrm{in}} \in Y_{\sigma,+}$, there is at least one global mild solution ψ to (1.5) in the sense of Definition 1.3 which belongs to $L^\infty((0,\infty), Y_{\sigma,+})$. Indeed, one applies Theorem 1.4 with $G_0(\zeta) = \zeta^\sigma$ and $\Lambda_i = i^\sigma$.

We next identify an additional simple structure condition on the kinetic coefficients $(\Gamma_{i,j})_{i,j\ge1}$ which, along with the boundedness of the fragment distribution function, allows us to show that the mild solution to (1.5a) constructed in Theorem 1.4 is actually a classical solution.

Theorem 1.5. *Assume that there is a non-negative sequence* $(\Lambda_i)_{i\ge1}$ *with* $\Lambda_1 \ge 1$ *such that the sequence* $(\Lambda_i/i)_{i\ge1}$ *is non-decreasing and the kinetic coefficients* $(\Gamma_{i,j})$ *satisfy*

$$0 \le \Gamma_{i,j} = \Gamma_{j,i} \le \Lambda_i\Lambda_j, \qquad i,j \ge 1. \tag{1.17}$$

Assume also that $(\varphi_{i,j;k})$ *satisfy the assumptions* (1.6) *and* (1.9) *with* $\alpha_1 = 0$*; that is,*

$$\varphi_{i,j;k} \le \alpha_0, \qquad 1 \le i \le j-1, \quad j \ge 2, \quad k \ge 1. \tag{1.18}$$

Finally, assume that the initial condition $\psi^{\mathrm{in}} \in Y_{1,+}$ *satisfies* (1.14) *for some function* $G_0 \in \mathcal{G}_{1,\infty}$*, as well as* (1.15)*. Then there is at least one mass conserving classical solution* $\psi = (\psi_i)_{i\ge1}$ *to* (1.5)*; that is, for each* $i \ge 1$, $\psi_i \in C^1([0,\infty))$,

$$(1-\delta_{i,1})\sum_{j=1}^{\infty} \Gamma_{i,j}\psi_i\psi_j \in C([0,\infty)), \qquad \sum_{j=i+1}^{\infty}\sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j\psi_k \in C([0,\infty)), \tag{1.19}$$

and (1.5a) *is satisfied pointwise for all* $i \ge 1$*. Furthermore,* ψ *satisfies*

$$M_\Lambda(\psi(t)) := \sum_{i=1}^{\infty} \Lambda_i\psi_i(t) \le M_\Lambda(\psi^{\mathrm{in}}), \qquad t \ge 0. \tag{1.20}$$

In contrast to (1.9), the assumption (1.18) implies the boundedness of the proportion of small clusters resulting from the collision of two clusters whatever their sizes, and thus precludes the formation of too many small clusters when a large cluster splits. The significant role of the structural assumption (1.17) in the existence theory for coagulation-fragmentation equations is already observed in [18,19]. Taking $\Lambda_i = \sqrt{A}i$, $i \geq 1$, the collision rate satisfies (1.7) and we recover the existence result established in [1, Theorem 2.1] by a different approach, but for a smaller set of initial conditions.

We supplement Theorem 1.5 with a uniqueness result under the assumption of the finiteness of a higher moment of the initial condition. The proof relies on an estimate on the difference of two solutions in a suitably chosen weighted space, a classical approach to uniqueness for coagulation-fragmentation equations, see [4, Section 8.2.5] and the references therein.

Theorem 1.6. *Assume that the hypotheses of Theorem 1.5 hold, and let $\psi^{\mathrm{in}} \in Y_{1,+}$ satisfy*

$$M_{\Lambda^2}(\psi^{\mathrm{in}}) := \sum_{i=1}^{\infty} \Lambda_i^2 \psi_i^{\mathrm{in}} < \infty. \tag{1.21}$$

Then there exists a unique classical solution $\psi = (\psi_i)_{i\geq 1}$ to (1.5) such that

$$M_{\Lambda^2}(\psi(t)) := \sum_{i=1}^{\infty} \Lambda_i^2 \psi_i(t) \leq M_{\Lambda^2}(\psi^{\mathrm{in}}) \quad \text{for all } t \geq 0. \tag{1.22}$$

The paper is organized as follows. Section 2 is dedicated to establishing the proof of Theorem 1.4, employing a compactness method and the approximation of the system (1.5) by finite systems of ordinary differential equations. Section 3 is devoted to establishing the existence of classical solutions, which is derived from Theorem 1.4 and appropriate moment estimates. The uniqueness of classical solutions is then dealt with in Section 4. In Section 5, we investigate the asymptotic behavior of solutions.

2. Existence of mild solutions

We fix $\psi^{\mathrm{in}} \in Y_{1,+}$ satisfying (1.14) for some $G_0 \in \mathcal{G}_{1,\infty}$. We then define

$$G_1(\zeta) = \frac{G_0(\zeta)}{\zeta}, \quad \zeta > 0, \qquad G_1(0) = 0,$$

and recall that the properties of G_0 imply that G_1 is non-negative, increasing, and concave. In addition,

$$\lim_{\zeta\to\infty} \zeta G_1'(\zeta) = \lim_{\zeta\to\infty} \frac{\zeta G_0'(\zeta) - G_0(\zeta)}{\zeta} = \infty, \tag{2.1}$$

since $G_0 \in \mathcal{G}_{1,\infty}$.

For $p \geq 3$, let us next introduce the truncated version of (1.5) as

$$\frac{d\psi_i^p}{dt} = \sum_{j=i+1}^{p} \sum_{k=1}^{p} \Gamma_{j,k} \varphi_{i,j;k} \psi_j^p \psi_k^p - (1-\delta_{i,1}) \sum_{j=1}^{p} \Gamma_{i,j} \psi_i^p \psi_j^p, \tag{2.2a}$$

$$\psi_i^p(0) = \psi_i^{\mathrm{in}}, \tag{2.2b}$$

where $i \in \{1, 2, \cdots, p\}$. We first report the well-posedness of (2.2) which is a classical consequence of the Cauchy-Lipschitz (or Picard-Lindelöf) theorem, together with the mass conservation (2.4).

Lemma 2.1. *Let $p \geq 3$. There is a unique solution $\psi^p = (\psi_i^p)_{1\leq i\leq p} \in C^1\big([0,\infty), [0,\infty)^p\big)$ to (2.2). For any sequence of non-negative real numbers $(\upsilon_i)_{i\geq 1}$, there holds*

$$\frac{d}{dt}\sum_{i=1}^{p} \upsilon_i \psi_i^p + \sum_{j=2}^{p}\sum_{k=1}^{p}\Big(\upsilon_j - \sum_{i=1}^{j-1}\upsilon_i\varphi_{i,j;k}\Big)\Gamma_{j,k}\psi_j^p\psi_k^p = 0. \tag{2.3}$$

In particular,

$$\sum_{i=1}^{p} i\psi_i^p(t) = \sum_{i=1}^{p} i\psi_i^{\text{in}} \le \|\psi^{\text{in}}\|_1, \qquad t\in[0,\infty). \tag{2.4}$$

Proof. As already mentioned, the local well-posedness of (2.2) and the componentwise non-negativity of the solution are straightforward consequences of the Cauchy-Lipschitz theorem, since the right-hand side of (2.2) is locally Lipschitz continuous and quasi-positive in $\mathbb{R}^p$. Next, a simple computation leads to the identity (2.3) which, in turn, gives the mass conservation (2.4) (with the choice $\upsilon_i = i$, $i\ge 1$) and thereby excludes finite time blowup of the solution. □

Exploiting (2.3) for a particular class of sequences $(\upsilon_i)_{i\ge1}$ provides addition information on ψ^p.

Lemma 2.2. *Let $p\ge 3$ and consider a sequence of non-negative real numbers $(\upsilon_i)_{i\ge1}$ such that $(\upsilon_i/i)_{i\ge1}$ is non-decreasing. Then, for $t\ge 0$,*

$$0 \le \sum_{i=1}^{p}\upsilon_i\psi_i^p(t) \le \sum_{i=1}^{p}\upsilon_i\psi_i^{\text{in}}, \tag{2.5a}$$

$$0 \le \int_0^t \sum_{j=2}^{p}\sum_{k=1}^{p}\left(\upsilon_j - \sum_{i=1}^{j-1}\upsilon_i\varphi_{i,j;k}\right)\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)\,ds \le \sum_{i=1}^{p}\upsilon_i\psi_i^{\text{in}}. \tag{2.5b}$$

Proof. Let $j\ge 2$ and $k\ge 1$. Since $(\upsilon_i)_{i\ge1}$ is a non-negative sequence such that $(\upsilon_i/i)_{i\ge1}$ is non-decreasing, we infer from (1.6) that

$$\begin{aligned}\upsilon_j - \sum_{i=1}^{j-1}\upsilon_i\varphi_{i,j;k} &= \upsilon_j - \sum_{i=1}^{j-1}\frac{\upsilon_i}{i}i\varphi_{i,j;k}\\ &\ge \upsilon_j - \frac{\upsilon_j}{j}\sum_{i=1}^{j-1}i\varphi_{i,j;k} = \upsilon_j-\upsilon_j = 0.\end{aligned}$$

Each term on the left hand side of (2.3) is therefore non-negative and we obtain (2.5a) and (2.5b) after integration with respect to time. □

Several estimates on the solutions to (2.2) can then be deduced from Lemma 2.2. Let us begin with a uniform control on the tail of the first moment.

Lemma 2.3. *For $p\ge 3$, $2\le r\le p$, and $t\ge 0$, we have*

$$\sum_{i=r}^{p} i\psi_i^p(t) \le \sum_{i=r}^{p} i\psi_i^{\text{in}} \le \sum_{i=r}^{\infty} i\psi_i^{\text{in}}.$$

Proof. We note that the sequence defined by

$$\upsilon_i = 0,\quad 1\le i\le r-1, \qquad \upsilon_i = i,\quad i\ge r,$$

is non-negative and $(\upsilon_i/i)_{i\geq 1}$ is non-decreasing. We then apply Lemma 2.2 with this sequence and deduce Lemma 2.3 from (2.5a). □

We next turn to estimates on the reaction terms.

Lemma 2.4. *For $p > i \geq 1$, there is $C_i := \mathcal{J}_0/[iG_1'(i+1)] > 0$ such that*

$$(1-\delta_{i,1}) \int_0^t \sum_{j=1}^p \Gamma_{i,j} \psi_i^p(s) \psi_j^p(s)\, ds \leq C_i, \quad t \geq 0, \tag{2.6a}$$

$$\int_0^t \sum_{j=i+1}^p \sum_{k=1}^p \varphi_{i,j;k} \Gamma_{j,k} \psi_j^p(s) \psi_k^p(s)\, ds \leq C_i, \quad t \geq 0. \tag{2.6b}$$

Proof. Since $(G_1(i))_{i\geq 1} = \Big(G_0(i)/i\Big)_{i\geq 1}$ is a non-decreasing sequence, we infer from (1.14) and Lemma 2.2 with $\upsilon_i = G_0(i)$, $i \geq 1$, that, for $t \geq 0$,

$$\sum_{i=1}^p G_0(i) \psi_i^p(t) \leq \mathcal{J}_0 \tag{2.7}$$

and

$$0 \leq \int_0^t \sum_{j=2}^p \sum_{k=1}^p \left(G_0(j) - \sum_{i=1}^{j-1} G_0(i) \varphi_{i,j;k} \right) \Gamma_{j,k} \psi_j^p(s) \psi_k^p(s)\, ds \leq \mathcal{J}_0,$$

which also reads, according to (1.6) and the definition of G_1,

$$0 \leq \int_0^t \sum_{j=2}^p \sum_{k=1}^p \sum_{i=1}^{j-1} (G_1(j) - G_1(i)) i \varphi_{i,j;k} \Gamma_{j,k} \psi_j^p(s) \psi_k^p(s)\, ds \leq \mathcal{J}_0. \tag{2.8}$$

Owing to the concavity of G_1,

$$G_1(j) - G_1(l) \geq (j-l) G_1'(j), \quad 1 \leq l \leq j-1, \tag{2.9}$$

so that, using once more (1.6), we have

$$\begin{aligned}
\sum_{i=1}^{j-1} \big(G_1(j) - G_1(i)\big) i \varphi_{i,j;k} &\geq \sum_{i=1}^{j-1} i(j-i) G_1'(j) \varphi_{i,j;k} \\
&= G_1'(j) \left(j^2 - \sum_{i=1}^{j-1} i^2 \varphi_{i,j;k} \right) \\
&\geq G_1'(j) \left(j^2 - (j-1) \sum_{i=1}^{j-1} i \varphi_{i,j;k} \right) \\
&= j G_1'(j).
\end{aligned}$$

Combining the above lower bound with (2.8), we find

$$\int_0^t \sum_{j=2}^p \sum_{k=1}^p j G_1'(j)\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)\,ds \le \mathcal{J}_0. \tag{2.10}$$

In particular, for $p \ge i \ge 2$ and $t > 0$, we deduce from (2.10) that

$$\mathcal{J}_0 \ge \int_0^t \sum_{k=1}^p i G_1'(i)\Gamma_{i,k}\psi_i^p(s)\psi_k^p(s)\,ds,$$

whence (2.6a) since $G_1'(i) \ge G_1'(i+1)$.

We next infer from (2.8), (2.9), and the monotonicity of G_1 that, for $p \ge i \ge 1$ and $t \ge 0$,

$$\begin{aligned}
\mathcal{J}_0 &\ge \int_0^t \sum_{j=i+1}^p \sum_{k=1}^p \sum_{l=1}^{j-1} (G_1(j)-G_1(l)) l \varphi_{l,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\ge \int_0^t \sum_{j=i+1}^p \sum_{k=1}^p (G_1(j)-G_1(i)) i \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\ge i[G_1(i+1)-G_1(i)] \int_0^t \sum_{j=i+1}^p \sum_{k=1}^p \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\ge i G_1'(i+1) \int_0^t \sum_{j=i+1}^p \sum_{k=1}^p \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds,
\end{aligned}$$

which completes the proof. □

An immediate consequence of Lemma 2.4 is the following bound on the time derivative of ψ_i^p.

Corollary 2.5. *For $p \ge i \ge 1$ and $t \ge 0$, we have*

$$\int_0^t \left| \frac{d\psi_i^p}{dt}(s) \right| ds \le 2C_i. \tag{2.11}$$

We now turn to the derivation of additional estimates on the reaction terms. Following the strategy outlined in [1, Proof of Theorem 2.1], we arrive at the main estimate of this section, which provides control over the contributions arising from the tails of the two infinite series on the right-hand side of (1.13).

Proposition 2.6. *For $i \ge 1$, $t > 0$ and $i \le m < p$,*

$$(1-\delta_{i,1}) \int_0^t \sum_{j=m+1}^p \Gamma_{i,j}\psi_i^p(s)\psi_j^p(s)ds \le \varepsilon_m, \tag{2.12}$$

$$\int_0^t \sum_{j=m+1}^p \sum_{k=1}^p \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \le \omega_m(i), \tag{2.13}$$

$$\int_0^t \sum_{j=i+1}^m \sum_{k=m+1}^p \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \le \alpha_0\varepsilon_m + \alpha_1\omega_m(i), \tag{2.14}$$

with

$$\varepsilon_m := \frac{\mathcal{J}_0}{\inf\limits_{z\ge m}\{zG_1'(z)\}} \quad \textit{and} \quad \omega_m(i) := \frac{\mathcal{J}_0}{G_1(m+1)-G_1(i)}. \tag{2.15}$$

Proof. Consider first $i \ge 2$ and $i \le m < p$. Then, by (2.10),

$$\begin{aligned}
\mathcal{J}_0 &\ge \int_0^t \sum_{j=2}^{p} jG_1'(j)\Gamma_{j,i}\psi_j^p(s)\psi_i^p(s)ds \\
&\ge \int_0^t \sum_{j=m+1}^{p} jG_1'(j)\Gamma_{i,j}\psi_i^p(s)\psi_j^p(s)ds \\
&\ge \inf_{z\ge m}\{zG_1'(z)\} \int_0^t \sum_{j=m+1}^{p} \Gamma_{i,j}\psi_i^p(s)\psi_j^p(s)ds,
\end{aligned}$$

and we have proved (2.12).

Next, for $1 \le i \le m < p$, we infer from (2.8) and the monotonicity of G_1 that

$$\begin{aligned}
\mathcal{J}_0 &\ge \int_0^t \sum_{j=m+1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{j-1}(G_1(j)-G_1(l))l\varphi_{l,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\ge \int_0^t \sum_{j=m+1}^{p}\sum_{k=1}^{p}(G_1(j)-G_1(i))i\varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\ge i[G_1(m+1)-G_1(i)] \int_0^t \sum_{j=m+1}^{p}\sum_{k=1}^{p}\varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds,
\end{aligned}$$

which gives (2.13). Now, by (1.9),

$$\begin{aligned}
\int_0^t \sum_{j=i+1}^{m}\sum_{k=m+1}^{p}\varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds &\le \alpha_0 \int_0^t \sum_{j=i+1}^{m}\sum_{k=m+1}^{p}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\quad + \alpha_1 \int_0^t \sum_{j=i+1}^{m}\sum_{k=m+1}^{p}\varphi_{i,k;j}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds.
\end{aligned}$$

We estimate each of the terms on the right-hand side separately and begin with the first one. Owing to the monotonicity of G_1,

$$\begin{aligned}
\alpha_0 \int_0^t \sum_{j=i+1}^{m}\sum_{k=m+1}^{p}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds &= \alpha_0 \int_0^t \sum_{j=i+1}^{m}\sum_{k=m+1}^{p}\frac{kG_1'(k)}{kG_1'(k)}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\le \frac{\alpha_0}{\inf\limits_{z\ge m}\{zG_1'(z)\}} \int_0^t \sum_{j=i+1}^{m}\sum_{k=m+1}^{p} kG_1'(k)\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds
\end{aligned}$$

$$\le \frac{\alpha_0}{\inf\limits_{z\ge m}\{zG_1'(z)\}}\int_0^t \sum_{j=1}^p\sum_{k=2}^p kG_1'(k)\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds.$$

Applying the substitution $j \leftrightarrow k$ and invoking (2.10), we obtain

$$\alpha_0\int_0^t \sum_{j=i+1}^m\sum_{k=m+1}^p \Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \le \frac{\alpha_0\mathcal{J}_0}{\inf\limits_{z\ge m}\{zG_1'(z)\}}.$$

We next deal with the second term and use once more the monotonicity of G_1 to obtain

$$\begin{aligned}
&\alpha_1\int_0^t \sum_{j=i+1}^m\sum_{k=m+1}^p \varphi_{i,k;j}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&= \alpha_1\int_0^t \sum_{j=i+1}^m\sum_{k=m+1}^p \frac{G_1(k)-G_1(i)}{G_1(k)-G_1(i)}\varphi_{i,k;j}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\le \frac{\alpha_1}{[G_1(m+1)-G_1(i)]}\int_0^t \sum_{j=i+1}^m\sum_{k=m+1}^p\sum_{l=1}^{k-1}[G_1(k)-G_1(l)]\varphi_{l,k;j}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \\
&\le \frac{\alpha_1}{[G_1(m+1)-G_1(i)]}\int_0^t \sum_{j=1}^p\sum_{k=2}^p\sum_{l=1}^{k-1}[G_1(k)-G_1(l)]l\varphi_{l,k;j}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds.
\end{aligned}$$

Applying estimate (2.8), we end up with

$$\alpha_1\int_0^t \sum_{j=i+1}^m\sum_{k=m+1}^p \varphi_{i,k;j}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \le \frac{\alpha_1\mathcal{J}_0}{[G_1(m+1)-G_1(i)]},$$

which completes the proof. □

Proof of Theorem 1.4. Owing to (2.4) and (2.11), the sequence $(\psi_i^p)_{p\ge3}$ is uniformly bounded and has uniformly bounded variation on every finite time interval for each $i \ge 1$. Consequently, we are in a position to apply Helly's selection principle [21, Theorem 2.35], along with a diagonal argument, to conclude that there exist a subsequence of $(\psi^p)_{p\ge3}$, still denoted by $(\psi^p)_{p\ge3}$, and a sequence of functions $\psi = (\psi_i)_{i\ge1}$ such that

$$\lim_{p\to\infty}\psi_i^p(t) = \psi_i(t) \quad \text{for all } t \ge 0 \text{ and } i \ge 1. \tag{2.16}$$

Clearly, $\psi_i(t) \ge 0$ for $i \ge 1$ and $t \ge 0$, and it follows from (2.4), Lemma 2.3, and (2.16) that, for $r \ge 1$, $p \ge q \ge r+1$ and $t \ge 0$,

$$\sum_{i=r}^q i\psi_i(t) = \lim_{p\to\infty}\sum_{i=r}^q i\psi_i^p(t) \le \sum_{i=r}^\infty i\psi_i^{\text{in}}.$$

By letting $q \to \infty$, we obtain

$$\sum_{i=r}^\infty i\psi_i(t) \le \sum_{i=r}^\infty i\psi_i^{\text{in}}, \quad r \ge 1,\ t \ge 0. \tag{2.17}$$

A similar argument allows us to deduce that

$$\sum_{i=1}^{\infty} G_0(i)\psi_i(t) \le \mathcal{J}_0, \quad t \ge 0, \tag{2.18}$$

from (2.7) and (2.16). Next, let $t \ge 0$, $i \ge 1$, and $r > i$. It follows from Lemma 2.4 and (2.16) that

$$(1-\delta_{i,1}) \int_0^t \sum_{j=1}^{r} \Gamma_{i,j}\psi_i(s)\psi_j(s)ds = (1-\delta_{i,1}) \lim_{p\to\infty} \int_0^t \sum_{j=1}^{r} \Gamma_{i,j}\psi_i^p(s)\psi_j^p(s)ds \le C_i$$

and

$$\int_0^t \sum_{j=i+1}^{r} \sum_{k=1}^{r} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s)ds = \lim_{p\to\infty} \int_0^t \sum_{j=i+1}^{r} \sum_{k=1}^{r} \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s)ds \le C_i.$$

Letting $r \to \infty$ and using Fatou's lemma, we obtain

$$(1-\delta_{i,1}) \int_0^t \sum_{j=1}^{\infty} \Gamma_{i,j}\psi_i(s)\psi_j(s)ds \le C_i, \quad t \ge 0, \tag{2.19}$$

and

$$\int_0^t \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s)ds \le C_i, \quad t \ge 0. \tag{2.20}$$

In the same way, we infer from Proposition 2.6 and (2.16) that, for $m \ge i \ge 1$,

$$(1-\delta_{i,1}) \int_0^t \sum_{j=m+1}^{\infty} \Gamma_{i,j}\psi_i(s)\psi_j(s)ds \le \varepsilon_m \tag{2.21}$$

and

$$\int_0^t \sum_{j=m+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s)ds \le \omega_m(i), \tag{2.22}$$

$$\int_0^t \sum_{j=i+1}^{m} \sum_{k=m+1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s)ds \le \alpha_0\varepsilon_m + \alpha_1\omega_m(i). \tag{2.23}$$

We are now ready to complete the proof of Theorem 1.4 and first observe that it follows from the estimates (2.19) and (2.20) that Definition 1.3(ii) is satisfied. Besides, the convergence (2.16) and the tail control (2.17) imply that, for any $t \ge 0$ and $p \ge r \ge 2$,

$$\begin{aligned}\|\psi^p(t) - \psi(t)\|_1 &\le \sum_{i=1}^{r-1} i|\psi_i^p(t) - \psi_i(t)| + \sum_{i=r}^{\infty} i\psi_i(t) + \sum_{i=r}^{p} i\psi_i^p(t) \\ &\le \sum_{i=1}^{r-1} i|\psi_i^p(t) - \psi_i(t)| + 2\sum_{i=r}^{\infty} i\psi_i^{\text{in}}.\end{aligned}$$

Consequently,

$$\limsup_{p\to\infty} \|\psi^p(t) - \psi(t)\|_1 \le 2\sum_{i=r}^{\infty} i\psi_i^{\mathrm{in}},$$

and we take the limit $r \to \infty$ in the above inequality and use the summability properties of ψ^{in} to conclude that

$$\lim_{p\to\infty} \|\psi^p(t) - \psi(t)\|_1 = 0. \tag{2.24}$$

In particular, we infer from (2.4) and (2.24) that, for all $t \ge 0$, we have

$$\|\psi(t)\|_1 = \lim_{p\to\infty} \|\psi^p(t)\|_1 = \lim_{p\to\infty} \|\psi^p(0)\|_1 = \|\psi^{\mathrm{in}}\|_1,$$

which shows that ψ satisfies the mass conservation property (1.3).

We next fix $i \ge 1$ and $t \ge 0$. On the one hand, it follows from (2.12), (2.16), and (2.21) that, for $i+1 \le m < p$, we have

$$\begin{aligned}
(1-\delta_{i,1}) \int_0^t & \left| \sum_{j=1}^{p} \Gamma_{i,j}\psi_i^p(s)\psi_j^p(s) - \sum_{j=1}^{\infty} \Gamma_{i,j}\psi_i(s)\psi_j(s) \right| ds \\
&\le \int_0^t \sum_{j=1}^{m} \Gamma_{i,j} |\psi_i^p(s)\psi_j^p(s) - \psi_i(s)\psi_j(s)| ds + \int_0^t \sum_{j=m+1}^{p} \Gamma_{i,j}\psi_i^p(s)\psi_j^p(s) ds \\
&\quad + \int_0^t \sum_{j=m+1}^{\infty} \Gamma_{i,j}\psi_i(s)\psi_j(s) ds \\
&\le \int_0^t \sum_{j=1}^{m} \Gamma_{i,j} |\psi_i^p(s)\psi_j^p(s) - \psi_i(s)\psi_j(s)| ds + 2\varepsilon_m,
\end{aligned}$$

recalling that ε_m is defined in (2.15). By virtue of (2.4), (2.16), and (2.17), together with the Lebesgue dominated convergence theorem, we may pass to the limit as $p \to \infty$ in the above inequality to obtain

$$\limsup_{p\to\infty} (1-\delta_{i,1}) \int_0^t \left| \sum_{j=1}^{p} \Gamma_{i,j}\psi_i^p(s)\psi_j^p(s) - \sum_{j=1}^{\infty} \Gamma_{i,j}\psi_i(s)\psi_j(s) \right| ds \le 2\varepsilon_m.$$

Since $G_0 \in \mathcal{G}_{1,\infty}$, the function G_1 satisfies (2.1), so that the right hand side of the above inequality converges to zero as $m \to \infty$. We thus let $m \to \infty$ in the above inequality to conclude that

$$\lim_{p\to\infty} (1-\delta_{i,1}) \int_0^t \left| \sum_{j=1}^{p} \Gamma_{i,j}\psi_i^p(s)\psi_j^p(s) - \sum_{j=1}^{\infty} \Gamma_{i,j}\psi_i(s)\psi_j(s) \right| ds = 0. \tag{2.25}$$

We next turn to the convergence of the first term on the right hand side of (2.2a). For $p > m \ge i+1$, we infer from (2.13), (2.14), (2.22), and (2.23) that

$$\begin{aligned}
&\int_0^t \left| \sum_{j=i+1}^{p} \sum_{k=1}^{p} \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s) - \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s) \right| ds \\
&\qquad \le \int_0^t \left| \sum_{j=i+1}^{m} \sum_{k=1}^{m} \varphi_{i,j;k}\Gamma_{j,k}\big(\psi_j^p(s)\psi_k^p(s) - \psi_j(s)\psi_k(s)\big) \right| ds \\
&\qquad\quad + \int_0^t \sum_{j=i+1}^{m} \sum_{k=m+1}^{p} \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s) ds \\
&\qquad\quad + \int_0^t \sum_{j=m+1}^{p} \sum_{k=1}^{p} \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s) ds \\
&\qquad\quad + \int_0^t \sum_{j=i+1}^{m} \sum_{k=m+1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s) ds \\
&\qquad\quad + \int_0^t \sum_{j=m+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s) ds \\
&\qquad \le \int_0^t \left| \sum_{j=i+1}^{m} \sum_{k=1}^{m} \varphi_{i,j;k}\Gamma_{j,k}\big(\psi_j^p(s)\psi_k^p(s) - \psi_j(s)\psi_k(s)\big) \right| ds \qquad (2.26) \\
&\qquad\quad + 2\big(\alpha_0\varepsilon_m + \alpha_1\omega_m(i)\big) + 2\omega_m(i).
\end{aligned}$$

On the one hand, it follows from (2.4), (2.16), (2.17) and the Lebesgue dominated convergence theorem that

$$\lim_{p\to\infty} \int_0^t \left| \sum_{j=i+1}^{p} \sum_{k=1}^{p} \varphi_{i,j;k}\Gamma_{j,k}(\psi_j^p\psi_k^p - \psi_j\psi_k)(s) \right| ds = 0. \tag{2.27}$$

On the other hand, it follows from the properties (1.12), (2.1) and (2.15) that

$$\lim_{m\to\infty} \varepsilon_m = \lim_{m\to\infty} \omega_m(i) = 0. \tag{2.28}$$

Consequently, using (2.27), we may take the limit $p \to \infty$ in (2.26) and find

$$\begin{aligned}
\limsup_{p\to\infty} \int_0^t \left| \sum_{j=i+1}^{p} \sum_{k=1}^{p} \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s) - \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s) \right| ds \\
\le 2\big(\alpha_0\varepsilon_m + \alpha_1\omega_m(i)\big) + 2\omega_m(i).
\end{aligned}$$

We next let $m \to \infty$ and deduce from (2.28) that

$$\lim_{p\to\infty} \int_0^t \left| \sum_{j=i+1}^{p} \sum_{k=1}^{p} \varphi_{i,j;k}\Gamma_{j,k}\psi_j^p(s)\psi_k^p(s) - \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s) \right| ds = 0. \tag{2.29}$$

Owing to (2.16), (2.25), and (2.29), we may pass to the limit as $p \to \infty$ in the integral formulation of the equation satisfied by ψ_i^p, thereby concluding that ψ_i satisfies (1.13). Moreover, since the integrands on the right-hand side of (1.13) belong to $L^1_{\mathrm{loc}}([0,\infty))$, it follows that $\psi_i \in C([0,\infty))$, and we have proved the existence part of Theorem 1.4.

Finally, we assume that the initial condition ψ^{in} satisfies (1.15). Let $r \geq 1$ and $t \geq 0$. Since the sequence defined by $\upsilon_i = 0$ for $1 \leq i \leq r-1$ and $\upsilon_i = \Lambda_i$ for $i \geq r$ satisfies the assumptions of Lemma 2.2, we infer from (2.5a) that, for $p > m > r$, the solution ψ^p to (2.2) satisfies

$$\sum_{i=r}^{m} \Lambda_i \psi_i^p(t) \leq \sum_{i=r}^{p} \Lambda_i \psi_i^p(t) \leq \sum_{i=r}^{p} \Lambda_i \psi_i^{\mathrm{in}} \leq \sum_{i=r}^{\infty} \Lambda_i \psi_i^{\mathrm{in}}.$$

Thanks to (2.16), we first take the limit as $p \to \infty$ and subsequently let $m \to \infty$ in the above inequality to derive (1.16) and complete the proof of Theorem 1.4. □

3. Existence of classical solutions

Proof of Theorem 1.5. In this section, we assume that the kinetic coefficients $(\Gamma_{i,j})$, $(\varphi_{i,j;k})$, and the initial condition ψ^{in} satisfy (1.17), (1.6), (1.18), (1.14), and (1.15), respectively. It follows from Theorem 1.4 that (1.5) admits a global mass-conserving mild solution satisfying (1.16). Our goal is to prove that the additional assumptions (1.17) and (1.18) ensure the continuity properties (1.19) as well as the C^1-regularity of ψ_i for each $i \geq 1$. To this end, for $(s,t) \in [0,\infty)^2$ and $m \geq i \geq 2$, we deduce from (1.3), (1.16) and (1.17) that

$$\begin{aligned}
\left| \sum_{j=1}^{\infty} \Gamma_{i,j} \psi_i(t)\psi_j(t) - \sum_{j=1}^{\infty} \Gamma_{i,j} \psi_i(s)\psi_j(s) \right| &\leq \sum_{j=1}^{\infty} \Gamma_{i,j} |\psi_i(t)\psi_j(t) - \psi_i(s)\psi_j(s)| \\
&\leq \sum_{j=1}^{m} \Gamma_{i,j} |(\psi_i\psi_j)(t) - (\psi_i\psi_j)(s)| + \Lambda_i \sum_{j=m+1}^{\infty} \Lambda_j [(\psi_i\psi_j)(t) + (\psi_i\psi_j)(s)] \\
&\leq \sum_{j=1}^{m} \Gamma_{i,j} |(\psi_i\psi_j)(t) - (\psi_i\psi_j)(s)| + 2\Lambda_i \|\psi^{\mathrm{in}}\|_1 \sum_{j=m+1}^{\infty} \Lambda_j \psi_j^{\mathrm{in}}.
\end{aligned}$$

Due to the continuity of ψ_j for all $j \geq 1$,

$$\limsup_{s\to t} \left| \sum_{j=1}^{\infty} \Gamma_{i,j} \psi_i(t)\psi_j(t) - \sum_{j=1}^{\infty} \Gamma_{i,j} \psi_i(s)\psi_j(s) \right| \leq 2\Lambda_i \|\psi^{\mathrm{in}}\|_1 \sum_{j=m+1}^{\infty} \Lambda_j \psi_j^{\mathrm{in}},$$

and, applying (1.15), we can let $m \to \infty$ in the above inequality to conclude that

$$\lim_{s\to t} \sum_{j=1}^{\infty} \Gamma_{i,j} \psi_i(s)\psi_j(s) = \sum_{j=1}^{\infty} \Gamma_{i,j} \psi_i(t)\psi_j(t). \tag{3.1}$$

Similarly, consider $(s,t) \in [0,\infty)^2$ and $m > i \geq 1$. By making use of (1.6), (1.15), (1.16), and (1.17), we obtain

$$\begin{aligned}
&\left| \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k} \Gamma_{j,k} \psi_j(t)\psi_k(t) - \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k} \Gamma_{j,k} \psi_j(s)\psi_k(s) \right| \\
&\qquad \leq \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k} \Gamma_{j,k} |(\psi_j\psi_k)(t) - (\psi_j\psi_k)(s)|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=i+1}^{m} \sum_{k=1}^{m} \varphi_{i,j;k}\Gamma_{j,k}|(\psi_j\psi_k)(t) - (\psi_j\psi_k)(s)| \\
&\quad + \sum_{j=i+1}^{m} \sum_{k=m+1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}[(\psi_j\psi_k)(t) + (\psi_j\psi_k)(s)] \\
&\quad + \sum_{j=m+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}[(\psi_j\psi_k)(t) + (\psi_j\psi_k)(s)|.
\end{aligned}
$$

Thanks to (1.16), (1.17), and (1.18),

$$
\begin{aligned}
&\sum_{j=i+1}^{m} \sum_{k=m+1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}[(\psi_j\psi_k)(t) + (\psi_j\psi_k)(s)] \\
&\qquad \le \alpha_0 \sum_{j=i+1}^{m} \sum_{k=m+1}^{\infty} \Lambda_j\Lambda_k[(\psi_j\psi_k)(t) + (\psi_j\psi_k)(s)] \\
&\qquad \le \alpha_0 \left(M_\Lambda(\psi(t)) \sum_{k=m+1}^{\infty} \Lambda_k\psi_k(t) + M_\Lambda(\psi(s)) \sum_{k=m+1}^{\infty} \Lambda_k\psi_k(s) \right) \\
&\qquad \le 2\alpha_0 M_\Lambda(\psi^{\mathrm{in}}) \sum_{k=m+1}^{\infty} \Lambda_k\psi_k^{\mathrm{in}}.
\end{aligned}
$$

In a similar way we can evaluate

$$
\sum_{j=m+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}[(\psi_j\psi_k)(t) + (\psi_j\psi_k)(s) \le 2\alpha_0 M_\Lambda(\psi^{\mathrm{in}}) \sum_{j=m+1}^{\infty} \Lambda_j\psi_j^{\mathrm{in}}.
$$

Gathering the above estimates, we finally obtain

$$
\begin{aligned}
&\left| \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(t)\psi_k(t) - \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s) \right| \\
&\qquad \le \sum_{j=i+1}^{m} \sum_{k=1}^{m} \varphi_{i,j;k}\Gamma_{j,k}|(\psi_j\psi_k)(t) - (\psi_j\psi_k)(s)| + 4\alpha_0 M_\Lambda(\psi^{\mathrm{in}}) \sum_{j=m+1}^{\infty} \Lambda_j\psi_j^{\mathrm{in}}
\end{aligned}
$$

and proceed as in the derivation of (3.1) to find

$$
\lim_{s\to t} \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s) = \sum_{j=i+1}^{\infty} \sum_{k=1}^{\infty} \varphi_{i,j;k}\Gamma_{j,k}\psi_j(t)\psi_k(t). \tag{3.2}
$$

The continuity properties (1.19) follow directly from (3.1) and (3.2). Moreover, applying Definition 1.3(iii) in conjunction with (1.19), we deduce that $\psi_i \in C^1([0,\infty))$ for all $i \ge 1$. Finally, the bound (1.20) is an immediate consequence of (1.16) with $r = 1$. □

4. Uniqueness of classical solutions

Proof of Theorem 1.6. To begin with, we observe that the non-negativity and monotonicity of $(\Lambda_i/i)_{i\ge 1}$ entail that

$$\frac{\Lambda_{i+1}^2}{i+1} = (i+1)\left(\frac{\Lambda_{i+1}}{i+1}\right)^2 \ge i\left(\frac{\Lambda_i}{i}\right)^2 = \frac{\Lambda_i^2}{i}, \qquad i \ge 1,$$

so that $(\Lambda_i^2/i)_{i\ge1}$ is also a non-negative and non-decreasing sequence. Moreover, $\Lambda_i^2 \ge \Lambda_1\Lambda_i \ge \Lambda_i$ for $i \ge 1$ with $\Lambda_1^2 \ge 1$, so that (1.17) implies that $(\Gamma_{i,j})$ also satisfies (1.17) with Λ_i^2 instead of Λ_i. Taking into account (1.21), we are in a position to apply Theorem 1.5 to deduce the existence of at least one mass-conserving global classical solution ψ to (1.5) which satisfies the estimate (1.22).

We now establish the uniqueness result stated in Theorem 1.6. To this end, we follow the classical approach to uniqueness for coagulation-fragmentation equations, see [4, Section 8.2.5] for instance and the references therein, and derive an estimate on the difference between two solutions in a suitably chosen weighted space. Such an approach is already used in [1, Corollary 3.1] with a power weight and we adapt it here in the more general setting of Theorem 1.6, the appropriate weight being the sequence $(\Lambda_i)_{i\ge1}$. It is worth noting that, as usual, the proof requires slightly stronger conditions on the collision coefficients compared to those assumed for the existence result.

Let us thus consider two classical solutions $\psi = (\psi_i)_{i\ge1}$ and $\phi = (\phi_i)_{i\ge1}$ to (1.5) satisfying (1.22) and set $H := \psi - \phi$. Then, for $i \ge 1$, H_i solves

$$\frac{dH_i}{dt} = \sum_{j=i+1}^{\infty}\sum_{k=1}^{\infty}\Gamma_{j,k}\varphi_{i,j;k}[\psi_k H_j + \phi_j H_k] - (1-\delta_{i,1})\sum_{k=1}^{\infty}\Gamma_{i,k}[\psi_k H_i + \phi_i H_k],$$

from which we deduce that

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{\infty}\Lambda_i|H_i| &= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\sum_{k=1}^{\infty}\Lambda_i\,\mathrm{sgn}(H_i)\varphi_{i,j;k}\Gamma_{j,k}\left[\psi_k H_j + \phi_j H_k\right] \\
&\quad - \sum_{i=2}^{\infty}\sum_{k=1}^{\infty}\Lambda_i\,\mathrm{sgn}(H_i)\Gamma_{i,k}\left[\psi_k H_i + \phi_i H_k\right] \\
&= \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\sum_{i=1}^{j-1}\Lambda_i\,\mathrm{sgn}(H_i)\varphi_{i,j;k}\Gamma_{j,k}\left[\psi_k H_j + \phi_j H_k\right] \\
&\quad - \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\Lambda_j\,\mathrm{sgn}(H_j)\Gamma_{j,k}\left[\psi_k H_j + \phi_j H_k\right] \\
&\le \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\sum_{i=1}^{j-1}\Lambda_i\varphi_{i,j;k}\Gamma_{j,k}\left[\psi_k|H_j| + \phi_j|H_k|\right] \\
&\quad + \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\Lambda_j\Gamma_{j,k}\left[-\psi_k|H_j| + \phi_j|H_k|\right].
\end{aligned}$$

Owing to (1.6) and the monotonicity of $(\Lambda_i/i)_{i\ge1}$, we obtain

$$\sum_{i=1}^{j-1}\Lambda_i\varphi_{i,j;k} \le \frac{\Lambda_j}{j}\sum_{i=1}^{j-1} i\varphi_{i,j;k} = \Lambda_j.$$

Applying this bound gives

$$\frac{d}{dt}\sum_{i=1}^{\infty}\Lambda_i|H_i| \le \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\Lambda_j\Gamma_{j,k}\left[\psi_k|H_j| + \phi_j|H_k|\right]$$

$$+\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\Lambda_j\Gamma_{j,k}\big[-\psi_k|H_j|+\phi_j|H_k|\big]$$
$$=2\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\Lambda_j\Gamma_{j,k}\phi_j|H_k|$$

Finally, we conclude from (1.17), (1.22), and the above inequality that

$$\frac{d}{dt}\sum_{i=1}^{\infty}\Lambda_i|H_i|\le 2\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\Lambda_j^2\Lambda_k\phi_j|H_k|\le 2M_{\Lambda^2}(\psi^{\mathrm{in}})\sum_{i=1}^{\infty}\Lambda_i|H_i|,$$

and an application of Gronwall's lemma completes the proof. □

5. Large time behavior

We now examine the asymptotic behavior of solutions in the long-time regime. As previously noted, the dynamics of this model ensures that clusters fragment solely into smaller components upon collision. This structural constraint suggests that, as $t\to\infty$, the system evolves toward a state where only 1-clusters persist.

Proposition 5.1. *Suppose that* $(\Gamma_{i,j})$ *and* $(\varphi_{i,j;k})$ *satisfy* (1.8), (1.6), *and* (1.9). *For any* $\psi^{\mathrm{in}}\in Y_{1,+}$ *satisfying* (1.14) *for some* $G_0\in\mathcal{G}_{1,\infty}$, *the mass-conserving global mild solution* ψ *to* (1.5) *constructed in Theorem 1.4 satisfies*

$$\lim_{t\to\infty}\|\psi(t)-\psi^{\infty}\|_1=0$$

for some limiting sequence $\psi^{\infty}=(\psi_i^{\infty})_{i\ge1}\in Y_{1,+}$. *Furthermore, if* $\Gamma_{i,i}>0$ *for some* $i\ge2$, *then* $\psi_i^{\infty}=0$.

Proof. The proof proceeds along the same lines as that of [20, Proposition 4.1], see also [26], and relies on the time monotonicity of truncated masses. Specifically, for each $p\ge1$, we multiply the i-th equation by i and sum from $i=1$ to p. After integration with respect to time, we obtain that the truncated mass

$$\Delta_p(t):=\sum_{i=1}^{p}i\psi_i(t),\qquad t\ge0,$$

satisfies

$$\Delta_p(t_2)=\Delta_p(t_1)+\int_{t_1}^{t_2}\sum_{j=p+1}^{\infty}\sum_{k=1}^{\infty}\sum_{i=1}^{p}i\varphi_{i,j;k}\Gamma_{j,k}\psi_j(s)\psi_k(s)\,ds\ge\Delta_p(t_1)\tag{5.1}$$

for $t_2\ge t_1\ge0$ and is thus non-decreasing with respect to time. Since the total mass is conserved, Δ_p is uniformly bounded with respect to time on $[0,\infty)$, hence admits a finite limit $\Delta_p^{\infty}\ge0$ as $t\to\infty$. Noticing that $\psi_1=\Delta_1$ and $\psi_i=\Delta_i-\Delta_{i-1}$ for $i\ge2$, we deduce the existence of limits $\psi_1^{\infty}:=\Delta_1^{\infty}$ and $\psi_i^{\infty}:=\Delta_i^{\infty}-\Delta_{i-1}^{\infty}$, $i\ge2$,

$$\lim_{t\to\infty}\psi_i(t)=\psi_i^{\infty},\qquad i\ge1,$$

and the limiting sequence $(\psi_i^{\infty})_{i\ge1}$ belongs to $Y_{1,+}$.

A further consequence of (1.6) and (5.1) is that $p\Gamma_{p+1,p+1}\psi_{p+1}^2$ belongs to $L^1(0,\infty)$. In particular, if $\Gamma_{p+1,p+1}>0$ for some $p\ge1$, the corresponding estimate forces $\psi_{p+1}^{\infty}=0$. □

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Appendix A. The daughter distribution function (1.11)

This last section is devoted to the study of the daughter distribution function $(\varphi_{i,j;k})$ defined in (1.11), which is unbounded and satisfies the condition (1.9), as claimed in Remark 1.1.

Lemma A.1. *The fragment distribution function*

$$\varphi_{i,j;k} = \frac{1}{2^i} \frac{j 2^{j-1}}{2^j - j - 1}, \qquad 1 \le i \le j-1, \quad j \ge 2, \quad k \ge 1,$$

defined in (1.11)*, satisfies* (1.9) *with* $\alpha_0 = \alpha_1 = 1$.

Proof. Introducing the function

$$\xi(z) := \frac{z 2^{z-1}}{2^z - z - 1}, \qquad z \ge 2,$$

we note that

$$\varphi_{i,j;k} = \frac{\xi(j)}{2^i}, \qquad 1 \le i \le j-1, \quad j \ge 2, \quad k \ge 1.$$

A tedious computation reveals that ξ is increasing on $[4, \infty)$ with

$$\xi(2) = 4 > \xi(3) = 3 > \xi(4) = \frac{32}{11}.$$

We claim that

$$\xi(j) \le 2 + \xi(k), \qquad 2 \le j \le k. \tag{A.1}$$

Indeed, (A.1) is obvious for $j \ge 4$ due to the above mentioned monotonicity of ξ on $[4, \infty)$. We next observe that

$$\xi(3) \le 1 + \xi(4) \le 1 + \xi(k), \qquad k \ge 5,$$

and

$$\xi(2) \le 1 + \xi(3) \le 2 + \xi(4) \le 2 + \xi(k), \qquad k \ge 5,$$

where we have used again the monotonicity of ξ on $[4, \infty)$ to derive the above inequalities for $k \ge 5$.

Having established (A.1), we consider $2 \leq j \leq k$ and $1 \leq i \leq j-1$. It follows from (A.1) that

$$\varphi_{i,j;k} = \frac{\xi(j)}{2^i} \leq \frac{2+\xi(k)}{2^i} \leq 1 + \varphi_{i,k;j},$$

which shows the validity of (1.9) with $\alpha_0 = \alpha_1 = 1$ and completes the proof. □

References

[1] M. Ali, A.K. Giri, Ph. Laurençot, Well-posedness of the discrete collision-induced breakage equation with unbounded fragment distribution, Nonlinear Anal., Real World Appl. 75 (2024) 103967.
[2] M. Ali, A.K. Giri, Ph. Laurençot, The discrete collision-induced breakage equation with mass transfer: well-posedness and stationary solutions, SIAM J. Math. Anal. 56 (3) (2024) 2915–2937.
[3] M. Ali, A.K. Giri, A note on the discrete coagulation equations with collisional breakage, Acta Appl. Math. 189 (5) (2024).
[4] J. Banasiak, W. Lamb, Ph. Laurençot, Analytic Methods for Coagulation-Fragmentation Models, CRC Press, Boca Raton, FL, 2019.
[5] P.K. Barik, F.P. da Costa, J.T. Pinto, R.S. Sasportes, The discrete generalized exchange-driven system, Math. Methods Appl. Sci. 49 (2026) 2007–2027.
[6] P.K. Barik, A.K. Giri, Global classical solutions to the continuous coagulation equation with collisional breakage, Z. Angew. Math. Phys. 71 (38) (2020) 1–23.
[7] P.K. Barik, A.K. Giri, Weak solutions to the continuous coagulation model with collisional breakage, Discrete Contin. Dyn. Syst., Ser. A 40 (11) (2020) 6115–6133.
[8] Z. Cheng, S. Redner, Kinetics of fragmentation, J. Phys. A, Math. Gen. 23 (1990) 1233–1258.
[9] M. Ernst, E. Pagonabarraga, The non-linear fragmentation equation, J. Phys. A, Math. Theor. 40 (2007) F331–F337.
[10] E. Esentürk, Mathematical theory of exchange-driven growth, Nonlinearity 31 (2018) 3460–3483.
[11] A.K. Giri, Ph. Laurençot, Weak solutions to the collision-induced breakage equation with dominating coagulation, J. Differ. Equ. 288 (2021) 690–729.
[12] A.K. Giri, Ph. Laurençot, Existence and non-existence for collision-induced breakage equation, SIAM J. Math. Anal. 53 (4) (2021) 4605–4636.
[13] A.K. Giri, R.G. Jaiswal, Ph. Laurençot, The continuous collision-induced nonlinear fragmentation equation with non-integrable fragment daughter distributions, J. Differ. Equ. 403 (2024) 235–271.
[14] R.G. Jaiswal, A.K. Giri, Mass-conserving weak solutions to the continuous nonlinear fragmentation equation in the presence of mass transfer, Nonlinear Anal., Real World Appl. 85 (2025) 104381.
[15] M. Kostoglou, A.J. Karabelas, A study of the nonlinear breakage equation: analytical and asymptotic solutions, J. Phys. A, Math. Gen. 33 (2000) 1221–1232.
[16] M. Kostoglou, A.J. Karabelas, A study of the collisional fragmentation problem using the gamma distribution approximation, J. Colloid Interface Sci. 303 (2006) 419–429.
[17] P.L. Krapivsky, E. Ben-Naim, Shattering transitions in collision-induced fragmentation, Phys. Rev. E 68 (2) (2003) 021102.
[18] Ph. Laurençot, Global solutions to the discrete coagulation equations, Mathematika 46 (2) (1999) 433–442.
[19] Ph. Laurençot, The Redner–ben-Avraham–Kahng cluster system without growth condition on the kinetic coefficients, Port. Math. 82 (2025) 341–356.
[20] Ph. Laurençot, D. Wrzosek, The discrete coagulation equations with collisional breakage, J. Stat. Phys. 104 (2001) 193–220.
[21] G. Leoni, A First Course in Sobolev Spaces, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009.
[22] A. Schlichting, The exchange-driven growth model: basic properties and longtime behavior, J. Nonlinear Sci. 30 (2020) 793–830.
[23] S. Si, A.K. Giri, Existence and non-existence for exchange-driven growth model, Nonlinearity 38 (2025) 125014.
[24] M.V. Smoluchowski, Drei Vorträge über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen, Phys. Z. 17 (1916) 557–599.
[25] M.V. Smoluchowski, Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen, Z. Phys. Chem. 92 (1917) 129–168.
[26] L. Zheng, Asymptotic behavior of solutions to the nonlinear breakage equations, Commun. Pure Appl. Anal. 4 (2005) 463–473.