Results in Control and Optimization 22 (2026) 100673

Contents lists available at ScienceDirect

Results in Control and Optimization

journal homepage: www.elsevier.com/locate/rico

A generalized Vidale–Wolfe model on existing and new customers advertising strategy in a segmented market

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ARTICLE INFO

Keywords:
Advertising
Optimal control problem
Market segmentation
Maximum principle
Parameter estimation

ABSTRACT

Business firms and companies are constantly adopting the concept of market segmentation, which plays an important role in the promotion of a product. Additionally, advertising is another component that strengthens the company's communication with customers. To have effective marketing strategies, the implementation of independent advertising strategies for each segmented market is important. This paper deals with an optimal control problem that aims to obtain a dynamic advertising policy for new customers as well as minimize the decay rate for existing customers in a segment-specific market. We will an derive explicit optimal dynamic advertising efforts policy using Pontryagin's maximum principle. The analysis gives a deep insight into how the advertising effort should be planned by the decision-makers, designing strategies that maximize long-term profitability while effectively controlling advertising costs. The effectiveness of the suggested strategy is supported by numerical examples, along with parameter estimation to estimate the value of certain parameters.

1. Introduction

Modernization has forced almost every business organization across the globe to revolutionize the industrial sector, forcing every individual to shift their mode of shopping from the traditional way to online platforms. Furthermore, the demand for goods and services has increased on a larger scale. Thus, to meet this demand, consistent monitoring of the behavior of the customer and demands and using it to formulate a smart marketing strategy is essential. This is just the tip of the iceberg: modernization also demands sustainable operations coupled with innovative business models, including various platforms and marketplaces, on-demand services, and collaboration in various streams of the business unit to prioritize customer satisfaction. Technological advancement paved the way for out-of-the-box thinkers and innovators. The onset of the pandemic and various other reasons also put online platforms in the picture. Thus, in light of contemporary business dynamics, it is imperative for industries to strategically formulate and implement robust marketing strategies to enhance sales and navigate the fiercely competitive global business landscape. Also, the necessity of integrative research in marketing and operations management has been highlighted by numerous researchers.

Marketing aims to increase profit by adding value and building the goodwill of the company, which is unachievable if one wants to continue operating with current goods and services. As the initial stages of market penetration are crucial for the subsequent dissemination of a product, the introduction of a new product into the market involves a substantive amount of risk for the company. A significant number of marketing initiatives concentrate on advertising strategies that support the official introduction of new products and positively influence the diffusion curve. New-product companies especially aim to achieve the overarching goal of

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https://doi.org/10.1016/j.rico.2026.100673
Received 2 September 2025; Received in revised form 8 February 2026; Accepted 12 February 2026
Available online 13 February 2026

informing potential customers about the special features of their products through advertising. Companies exhibit a systematic approach to designing well-thought-out advertising strategies during new product launches. Also, maintaining a reliable connection with their current customers is of utmost importance for companies to maximize the total profit. For a detailed study of optimal advertising strategies, multiple research articles by the authors, including Sethi (1975) [1] and Little (1979) [2], provide in-depth insights into the finer points of advertising practices. Similarly, Hartl and Kort [3] claimed that dynamic advertising models represent one of the initial applications of Pontryagin's maximum principle in the field of economics and management, highlighting an innovative connection of theoretical principles with practical applications. Sethi (1977) [4] presented the first detailed review of the dynamic advertising literature, which was further updated by Feichtinger et al. (1994) [5]. But before introducing the product in the market, a business must determine which particular market segments are most likely to purchase the new product and must target the particular group accordingly. Current global marketplaces attract multicultural customers with a diverse spectrum of demands, preferences, and aspirations. Meeting the needs of every customer by treating them fairly remains extremely tough and unpredictable, even with the great efforts made in creativity and advertising planning. Therefore, to achieve this goal, businesses must segment a specific market into discrete groups, and marketing strategies are created according to these divisions. This signifies the need for division of markets into several market segments, each composed of customers who share the same demand characteristics.

Market segmentation [6] is the division of a market into distinct groups of customers that have similar demands/needs and product/service requirements. It is a marketing approach that entails segmenting a large target market into subsets of customers, firms, or countries that have or are seen to have common requirements, interests, and goals, and then devising and implementing tactics to target them. Market segmentation strategies are commonly used to identify and further define target customers as well as give supporting data for marketing plan parts such as positioning to achieve certain marketing plan objectives. Some of the variables play a major role in market segmentation, which includes geographic variables (such as regions, nations, countries, cities, etc.), demographic variables (such as gender, age, family size, income, occupation, etc.), psychographic variables (such as lifestyle, social class, personality, value, etc.), and behavioral variables (such as usage rate, user states, purchase occasion, and attitude towards product, etc.). Every consumer segment that has relatively identical demands will probably react to a certain marketing strategy similarly, helping firms to better comprehend and meet their consumers' demands. Following the segmentation of the market, firms identify diverse segments and formulate a marketing mix strategy for each identified segment. Businesses simultaneously employ segment-specific advertising techniques and mass-market advertising methods that target various segments within a particular spectrum.

The necessity of market segmentation and market initiatives has also been stressed by many researchers [7–11] in marketing literature. A model for creating advertising policies in a segmented market with several media possibilities has been brought out by Little and Lodish (1969) [12]. Seidman et al. (1987) [13] analyzed a sales and advertising model in which the population is split into segments and advertising is done at various levels. In the advertising models of Buratto et al. (2006a, 2006b) [14,15], market segmentation concepts are integrated with Nerlove and Arrow (1962) [16] linear goodwill dynamics to solve the introduction of a new product and the advertising channel selection problem in a segmented market. Jha et al. [17] used market segmentation as the basis for a diffusion model that took into account product advertising to investigate the ideal rate of advertising effectiveness in a segmented market. Using Nerlove–Arrow's linear goodwill dynamics in a segmented market, Favaretto and Viscolani (2010) [18] investigated an optimal control model based on production and advertising for a seasonal product. An innovation diffusion model was employed by Mehta, Chaudhary, and Kumar (2020) [19] to determine the best marketing strategies to adopt when a new product is introduced in a market segment with a fluctuating market size. Ma and Jiang (2022) [20] examined a single-parameter sales promotion approach in their advertising model and examined the stability of their suggested model. A model was presented by Kumar et al. (2022) [21] to identify the best combination of pricing and advertising to maximize integrated profit for a product in a segmented market with price-dependent market potential. Chaudhary et al. (2022) [22] have tackled the optimal control problem to identify the best promotional policies for a diffusion model in a segmented market based on the assumption that the goodwill of the firm and improved brand image will increase demand for the new product.

A fundamental component of modern marketing strategy is the multifaceted role of advertising. Companies allocate a significant portion of their promotional budgets not only to attract potential customers but also to retain and re-engage their existing ones. Advertising aimed at new customers enhances product awareness, disseminates information, and encourages initial adoption, embodying the traditional function of advertising within diffusion models. However, in highly competitive markets characterized by frequent switching behavior and brand substitution, retaining existing customers has become equally critical. Such advertising initiatives help maintain brand loyalty, mitigate the decline in product goodwill, and foster repeat purchase behavior, thereby ensuring long-term profitability.

Despite the extensive literature on dynamic advertising, most Vidale–Wolfe–based models still consider only a single aggregated advertising effort directed towards new customers and assume a constant decay rate independent of advertising directed solely for retaining new customers. This overlooks the strategic reality that firms invest substantially in retention to reduce churn and sustain profit. Moreover, although market segmentation is widely recognized as a key component of modern marketing, only a limited number of dynamic models incorporate segment-specific behaviors and advertising responses. Consequently, current quantitative approaches provide limited guidance on how firms should balance acquisition and retention efforts across diverse customer groups to maximize overall performance.

Recent research by Shiva et al. (2024) [23] introduces a change-point advertising expenditure model to examine its impact on product sales dynamics. Ma and Jiang (2022) [20], in their study investigate a single-parameter sales promotion strategy within a differential advertising framework, highlighting nonlinear and bifurcation behaviors. Although these studies capture clearly the shifts in advertising effectiveness and provide valuable understanding of advertising-driven sales evolution, they do not explicitly account for advertising targeted at retaining existing customers, at the same time treating all customers as a homogeneous group. In parallel, Chen, Zhu, and Sheng (2021) [24] analyze an optimal control advertising model under conditions of uncertainty, while Li and Xiao (2025) [25] extend the single-parameter advertising approach into a continuous differential framework, revealing complex dynamic outcomes. Collectively, these contributions reveal a persistent reliance of the existing literature on single advertising strategies, which motivates the need for development of the dual-control framework, which is proposed in this study.

In practical marketing applications, a clear distinction is often made between advertising strategies aimed at acquiring new customers and those focused on retaining existing ones. Coca-Cola, for instance, utilizes their "Share a Coke" advertising campaign to create an emotional connection between Coca-Cola and its existing customers (retain) and new customers via (in part) personalizing customer interactions with the brand [26]. Similarly, Apple is able to target their advertising campaign, "Shot on iPhone", to prospects (acquiring new customers) since it highlights the capabilities of their product through customer-generated demonstrations of photographs taken with an iPhone [27]. The example of Coca-Cola's "Share a Coke" campaign and Apple's "Shot on iPhone" advertising reveal that there are, in fact, different advertising strategies targeted toward acquisition and retention, with each strategy serving distinct strategic objectives. It is necessary to quantify both acquisition and retention strategies to present them in terms of how both advertising approaches to customers interact dynamically within segmented markets. Empirical research substantiates that user-generated content and traditional media have distinct impacts on the dynamics of customer acquisition and retention [28].

To address these limitations, we extend the Vidale–Wolfe framework in this paper by dealing with two forms of advertising controls: those aimed at maintaining existing customers and those directed toward acquiring new customers. The advertising strategy for existing customers aims to maintain their engagement and allegiance to a product, ensuring they remain connected and less likely to drift towards competing brands. Conversely, advertising strategies directed towards new customers employ a distinct approach, emphasizing comprehensive information dissemination about the product's core features and advantages. This approach seeks to attract and persuade potential new customers to adopt the product in different segments. Even though segmentation is one of the most important market strategies and practices, not much work has been done on dynamic quantitative advertising models with market segmentation. We will use the concept of market segmentation in the Vidale–Wolfe model to decrease the decay rate of existing customers and study the optimal advertising effectiveness rate of both new and existing customers towards a segmented market.

In light of the preceding discussions and the identified gaps in the literature, this study aims to address the following research questions:

- **RQ1:** How can an optimal control model be developed to maximize profit by simultaneously incorporating advertising efforts directed toward new customer acquisition and existing customer retention over time?
- **RQ2:** What method can be utilized to analyze the optimal advertising strategy in a market in a dynamic scenario?
- **RQ3:** How do varying key model parameters influence market share in a segmented market?
- **RQ4:** To what extent does the proposed model improve predictive performance and managerial relevance when validated using real sales data, in comparison with the classical Vidale–Wolfe model?

The key contributions of this study are highlighted as follows:

- We have extended the classical Vidale–Wolfe advertising response model by differentiating between advertising aimed at new customers and retention advertising aimed at current customers, thereby aligning more closely with contemporary marketing practices.
- We incorporate market segmentation into the model to account for diverse adoption behaviors among various customer groups, thereby enabling the firm to allocate advertising efforts in a manner specific to each segment.
- In contrast to existing studies on optimal advertising based on the Vidale–Wolfe model, which typically assume a constant decay rate, our approach allows the decay rate to be influenced by retention advertising. This enables us to model the impact of loyalty-enhancing efforts on reducing customer attrition.
- We formulate and examine an optimal control problem to ascertain the most effective advertising strategy for each market segment, deriving analytical expressions for the optimal policies within a quadratic promotional cost framework.
- We conduct parameter estimation utilizing real data in order to validate the applicability and superior predictive performance of the proposed model in comparison to the classical Vidale–Wolfe model.

These innovative extensions offer a more realistic and managerial decision-oriented framework for designing advertising policies in segmented markets.

The rest of the paper is structured as follows. Section 2 states the Vidale–Wolfe model for a monopolistic firm. Section 3 gives the model formulation followed by the proposed optimal control formulation in Section. Section 4 provides a numerical illustration of the proposed model. Finally, the managerial implications and conclusion of the proposed study are given in Section 5 and Section 6, respectively.

2. Vidale–Wolfe model

A number of advertising response models have been discussed in the literature for modeling an optimal advertisement policy for the product [29]. The purpose of such models is to depict the successive increase in the number of purchasers and to predict the continued development of the buying process already in progress. Models are used successfully by various researchers to evaluate the market response over the product life cycle and make valuable decisions related to product modification, price differentiation, resource optimization, etc. One of the commonly known and used advertising response models studied in this field is Vidale and Wolfe (1957) [30], as it represents the connection between advertising and sales in an intuitively satisfying manner.

According to Vidale and Wolfe, changes in the sales rate of a product are determined by two factors: response to advertising that affects the unsold segment of the market (by using the response constant β), and loss as a result of forgetting, which affects the sold part of the market (through the decay constant δ). Let $S(t)$ denote the actual sales rate at time t and $u(t)$ be the advertising expenditure rate at time t. Let $\overline{N}$ denote the maximum potential rate or saturation level of sales at time t. Hence, the Vidale–Wolfe model for the monopolistic firm can be represented in the form of a differential equation:

$$\frac{dS(t)}{dt} = \beta u(t)\left(1 - \frac{S(t)}{\overline{N}}\right) - \delta S(t). \tag{1}$$

The Vidale and Wolfe model, originally intended to describe market phenomena, can be adapted to find the ideal advertising expenditures that maximize an objective function over a time horizon T while achieving a target sales goal [31]. The optimal control problem is defined as:

$$\begin{cases} Max.J = \int_0^T e^{-\rho t}\big(\pi x - u\big)dt, \\ \text{The terminal state constraint,} \quad x(T) = x_T. \end{cases} \tag{2}$$

Here, $'J'$ is the profit function, and $'u'$ is the control constraint, $0 \le u \le Q$. Here, Q can be finite or infinite, and the target market share $x_T \in [0, 1]$.

3. Model formulation

In this work, we expand on the Vidale and Wolfe model by differentiating between two types of advertising efforts: unsold market segments (new consumers) and declining sales (current customers). Existing consumers are targeted in advertising efforts to keep them from forgetting about the product and switching to rival brands. Advertising targeting new consumers is unique in that it provides more of the underlying product in order to persuade new customers. We concentrate solely on the sales and marketing models. Considering the effects of both types of advertising, we extended the V–W model by using a differential system:

$$\frac{dS(t)}{dt} = \beta u_1(t)\left(1 - \frac{S(t)}{\overline{N}}\right) - \delta(u_2)S(t). \tag{3}$$

In the classical Vidale–Wolfe model, the parameter δ denotes the decay rate of sales or adoption in the absence of advertising, encapsulating the empirical observation that the impact of advertising "lingers on but diminishes with time" [4,30]. In the absence of advertising, customers may lose interest or switch to competitors, causing sales to decay over time. Retention advertising (u_2) is designed to mitigate this decay by reminding existing customers about the product and reinforcing loyalty. Thus, $\delta(u_2)$ is assumed to be a decreasing function of u_2, i.e., stronger retention efforts cannot increase churn.

To improve practical understanding of the proposed model, we have included the managerial meaning of the parameters used in the analysis which is presented Table 1 below.

3.1. Proposed optimal control formulation

Optimal control theory provides a robust framework for determining the temporal variation of advertising efforts to enhance long-term performance. In marketing contexts, customer acquisition and retention are dynamic processes, and current advertising decisions impact future profitability. Consequently, a time-dependent optimization approach is necessary to balance the immediate benefits of increased sales with the long-term advantages of maintaining the customer base [32].

Let $u_{1i}(t)$ and $u_{2i}(t)$ be the advertising effectiveness rates for new and existing customers, respectively, at time t towards segment i, and this rate influences the remaining unsold segment of the market $(\overline{N}_i - S_i(t))$, and δ_i is the decay rate that impacts the sold segment of the market. Hence, the differential equation representing the rate of sales intensity is as follows:

$$\frac{dS_i(t)}{dt} = \beta_i u_{1i}(t)\left(1 - \frac{S_i(t)}{\overline{N}_i}\right) - \delta_i(u_{2i})S_i(t), \tag{4}$$

To establish the optimal control problem derived from the proposed model within a segmented market, we modify Eq. (4) through the substitution of $x_i(t) = \frac{S_i(t)}{\overline{N}_i}$ and $r_i = \frac{\beta_i}{\overline{N}_i}$, where x_i (t) represents the market share (the rate of sales expressed as a fraction of the saturation level $\overline{N}_i$) and the evolution of sales is defined by the differential equation $\frac{dx_i}{dt}$. Also, r_i represents the rate at which advertising converts the remaining potential market into adopters, expressed as a fraction of the segment market size. We can rewrite Eq. (4) as follows:

$$\frac{dx_i(t)}{dt} = r_i u_{1i}(t)\big(1 - x_i(t)\big) - \delta_i(u_{2i})x_i(t), \quad x_i(0) = x_{i0}, \quad x_i(t) \in [0, 1]. \tag{5}$$

Table 1
Interpretation of model parameters.

Parameter	Meaning
$\overline{N}_i$	Maximum market potential (size of segment i)or saturation level of sales at time t.
x_i	Market share (the rate of sales expressed as a fraction of the saturation level $\overline{N}_i$).
r_i	Advertising responsiveness of new customers in segment i (conversion efficiency).
α_i	Natural decay (churn/forgetting) rate of existing customers without retention advertising.
γ_i	Effectiveness of retention advertising in reducing the churn rate.
π_i	Per-unit profit margin from customers in segment i.
ρ_i	Discount factor reflecting the importance of short-term vs. long-term profit.
ϵ_{1i}	Cost-sensitivity parameter for advertising directed at new customers (higher value $\Rightarrow$ more expensive acquisition).
ϵ_{2i}	Cost-sensitivity parameter for advertising directed at existing customers (higher value $\Rightarrow$ more expensive retention).

Eq. (5) represents the modified Vidale–Wolfe's advertising model for a particular segment.

The media advertisement effectiveness is determined by how well it meets the company's profit or sales targets. As a result, the following optimal control problem is used to formulate the problem of finding out the optimum advertising effectiveness rates $u_{1i}(t)$ and $u_{2i}(t)$ for the new and existing customers, respectively, such that the total discounted profit over the planning horizon is maximized.

$$J[u_{1i},u_{2i}] = \int_0^T \sum_{i=1}^{M} e^{-\rho_i t}\Big(\pi_i x_i(t) - C_{1i}\big(u_{1i}(t)\big) - C_{2i}\big(u_{2i}(t)\big)\Big)dt. \tag{6}$$

subject to

$$\frac{dx_i}{dt} = r_i u_{1i}(t)\Big(1-x_i(t)\Big) - \delta_i(u_{2i})x_i(t), \quad x_i(0)=x_{i0}.$$

Where $\pi_i > 0$ represents the unit profit margin obtained from one unit of sale in segment i. The parameter $\rho_i > 0$ denotes the discount rate reflecting the time value of profit, $C_{1i}(u_{1i})$ and $C_{2i}(u_{2i})$ are instantaneous advertising cost functions for new and existing customers, respectively, towards the ith segment, and we assume that for all $i \in M$, $C_{1i}(u_{1i})$ and $C_{2i}(u_{2i})$ are two continuous differentiable functions of $u_{1i} \geq 0$ and $u_{2i} \geq 0$, respectively, such that

$$\frac{\partial C_{1i}}{\partial u_{1i}} \geq 0 \;\;\&\;\; \frac{\partial C_{2i}}{\partial u_{2i}} \geq 0 \;\;\&\;\; \frac{\partial^2 C_{1i}}{\partial^2 u_{1i}} \geq 0 \;\;\&\;\; \frac{\partial^2 C_{2i}}{\partial^2 u_{2i}} \geq 0.$$

Eq. (6) can be written as

$$J = \sum_{i=1}^{M} J_i,$$
$$\Rightarrow J_i = \int_0^T e^{-\rho_i t}\Big(\pi_i x_i(t) - C_{1i}\big(u_{1i}(t)\big) - C_{2i}\big(u_{2i}(t)\big)\Big)dt. \tag{7}$$

J_i is the profit density function of segment $i \in M$, which represents the optimal control problem for one state variable, i.e., $x_i(t)$, and two control variables, i.e., u_{1i} and u_{2i}.

Therefore, the segment dependent product introduction problem is equivalent to determining it for all $i \in M$ and advertising effectiveness rates $u_{1i}(t) \geq 0$ and $u_{2i}(t) \geq 0$ with the associated sales rate, $x_i(t)$, satisfying Eq. (5).

Hamiltonians can be defined by the Maximum Principle as,

$$H = \Big[\pi_i x_i(t) - C_{1i}\big(u_{1i}(t)\big) - C_{2i}\big(u_{2i}(t)\big)\Big] + \lambda_i\Big(r_i u_{1i}(t)\big(1-x_i(t)\big) - \delta_i(u_{2i})x_i(t)\Big). \tag{8}$$

Hamiltonians can be simply understood as the total profit from all policy choices, taking into consideration both the short- and long-term effects. Assuming the existence of an optimal control solution, the maximum principle provides the necessary optimality conditions [33,34]. There will exist an adjoint variable $\lambda_i(t)$ for all $t \in [0,T]$, which is a piecewise continuous differentiable function. The value of $\lambda_i(t)$ at time t describes the future effects on profits of making a small change in $x_i(t)$ at time t. It shows the similar behavior in optimal control theory as dual variables have in linear programming. For necessary conditions, we have

$$\frac{\partial H_i}{du_{1i}} = 0, \quad \text{and} \quad \frac{\partial H_i}{du_{2i}} = 0, \tag{9}$$

$$\frac{\partial \lambda_i}{\partial t} = \rho_i \lambda_i - \frac{\partial H_i}{\partial x_i}, \qquad \lambda_i(T) = 0. \tag{10}$$

From Eq. (9), we have

$$\frac{\partial H_i}{du_{1i}} = -\frac{\partial C_{1i}}{\partial u_{1i}} + \lambda_i r_i \Big(1 - x_i(t)\Big) = 0, \tag{11}$$

$$\frac{\partial H_i}{du_{2i}} = -\frac{\partial C_{2i}}{\partial u_{2i}} + \lambda_i \left(-\frac{\partial \delta_i}{\partial u_{2i}} x_i(t)\right) = 0. \tag{12}$$

Therefore, the Hamiltonian for each segment is strictly concave in u_{1i} and u_{2i}. By using the Mangasarian sufficiency theorem, we will be able to obtain unique values of advertising effectiveness rates for new and existing customers, i.e., u^*_{1i} and u^*_{2i}.

$$u^*_{1i} = C^{-1}_{1i}\Big(\lambda_i r_i \big(1 - x_i(t)\big)\Big), \tag{13}$$

and $$u^*_{2i} = C^{-1}_{2i}\big(\lambda_i \gamma_i x_i(t)\big). \tag{14}$$

Where C^{-1}_{1i} is the inverse function of the marginal cost rate $C_{u_{1i}}$, and C^{-1}_{2i} is the inverse function of the marginal cost rate $C_{u_{2i}}$. The optimal advertising effectiveness rates for new and existing customers should be zero or minimum, i.e., $u^*_{1i} = 0$ and $u^*_{2i} = 0$, when the market is almost saturated, i.e., $x_i(t) \to 1$. The co-state (adjoint) variable for optimal control policy is defined as

$$\lambda_i(t) = \pi_i e^{\int_t^T h(\tau)d\tau} \int_t^T e^{\int_t^T h(s)ds} dt. \tag{15}$$

where $h(\tau) = \rho_i + r_i u_{1i} + \delta(u_{2i})$.

The above Eq. (15) represents the future profit associated with obtaining one additional unit of sales or, equivalently, marginal value per unit of sale at time t.

3.2. Subclass of model formulation

The general formulation of the proposed model is covered in the section above. Even while the general proposed model formulation is helpful in understanding the aspects influencing the best possible advertising efforts, there are specific cases where we might achieve better outcomes. Let us assume that advertising cost is quadratic in u_{1i} and u_{2i}, i.e., $C_{1i}(u_{1i}) = \frac{\epsilon_{1i}}{2} u^2_{1i}(t), C_{2i}(u_{2i}) = \frac{\epsilon_{2i}}{2} u^2_{2i}(t)$, where $\epsilon_{1i} > 0$ and $\epsilon_{2i} > 0$ are positive constraints and capture the marginal cost of advertising effort in the ith segment. Quadratic cost functions are commonly adopted in the advertising and innovation diffusion fields due to the ability to maintain analytical tractability and the ability to capture the increasing marginal cost of advertising effort. This case can be seen in prior studies like Teng and Thompson [35], and more recently, Chen and Guo [36], wherein the cost of advertising is an increasing convex function of advertising effort.

In practical applications, firms allocate a significant portion of their advertising budget to existing customers (e.g., loyalty campaigns, reminder advertising, and relationship marketing) specifically to mitigate this decay by reducing forgetting and brand switching. To reflect this retention effect, we assume that the effective decay rate $\delta(u_2)$ can be reduced by advertising efforts directed toward existing customers. Thus, the decay rate $\delta(u_2)$ is a decreasing function in u_2, i.e., $\delta' \leq 0$, so that stronger retention advertising never increases the decay rate and produces diminishing marginal improvements in customer retention. As a simple specification consistent with the linear Vidale–Wolfe structure, we consider the function $\delta(u_2) = \alpha - \gamma u_2, \quad \alpha > 0,\ \gamma > 0$, where α_i is the natural decay rate of existing customers and γ_i measures how effectively retention advertising (u_2) reduces this decay.

Substituting $\delta(u_2)$ and using Eq. (6), we will then formulate the optimal control problem as:

$$J[u_{1i}, u_{2i}] = \int_0^T \sum_{i=1}^{M} e^{-\rho_i t}\Big(\pi_i x_i(t) - \frac{\epsilon_{1i}}{2} u^2_{1i}(t) - \frac{\epsilon_{2i}}{2} u^2_{2i}(t)\Big) dt. \tag{16}$$

subject to

$$\frac{dx_i}{dt} = r_i u_{1i}(t)\Big(1 - x_i(t)\Big) - \big(\alpha_i - \gamma_i u_2\big) x_i(t), \quad x_i(0) = x_{i0}.$$

By substituting $C_{1i} = \frac{\epsilon_{1i}}{2} u^2_{1i}(t)$ and $C_{2i} = \frac{\epsilon_{2i}}{2} u^2_{2i}(t)$ in Eq. (8), we will get the Hamiltonian function as:

$$H = \Big[\pi_i x_i(t) - \frac{\epsilon_{1i}}{2} u^2_{1i}(t) - \frac{\epsilon_{2i}}{2} u^2_{2i}(t)\Big] + \lambda_i\Big(r_i u_{1i}(t)\big(1 - x_i(t)\big) - \big(\alpha_i - \gamma_i u_{2i}\big) x_i(t)\Big). \tag{17}$$

Using necessary conditions,

$$\frac{\partial H_i}{du_{1i}} = -\epsilon_{1i} u_{1i} + \lambda_i r_i \Big(1 - x_i(t)\Big) = 0, \tag{18}$$

$$\frac{\partial H_i}{du_{2i}} = -\epsilon_{2i} u_{2i} - \lambda_i x_i(t)(-\gamma_i) = 0. \tag{19}$$

Using Eqs. (18) and (19) and Maximum Principle, we get the optimal advertising effectiveness rates as

$$u^*_{1i} = \frac{\lambda_i r_i \big(1 - x_i(t)\big)}{\epsilon_{1i}}, \tag{20}$$

Table 2
Data description.

Data sets (Dataset-I) Year	Sales	(Dataset-II) Year	Subscriptions	(Dataset-III) Year	Sales
March'07	1601.28	2005	90,140,000	2012	5032.72
"'08	3834.32	2006	166,050,000	2013	5670.96
"'09	6403.41	2007	233,620,000	2014	6342.21
"'10	8952.84	2008	346,890,000	2015	7263.52
"'11	11,841.41	2009	525,090,000	2016	8080.45
"'12	14,541.69	2010	752,190,000	2017	8829.17
"'13	17,282.24	2011	893,862,000	2018	9536.10
"'14	20,033.44	2012	864,721,000	2019	10,672.97
"'15	23,072.92	2013	886,304,000	2020	11,322.11
"'16	26,567.73	2014	944,009,000	2021	12,671.53
"'17	30,679.82	2015	1,001,060,000	2022	13,731.05
"'18	35,095.79	2016	1,127,810,000	2023	15,839.01
"'19	39,879.49	2017	1,168,900,000	2024	16,186.08
"'20	44,665.98	2018	1,176,020,000	2025	17,295.92
March'21	48,508.21	2019	1,151,480,000		
		2020	1,153,710,000		
		2021	1,154,050,000		
		2022	1,142,930,000		
		2023	1,158,550,000		

and $$u_{2i}^* = \frac{\lambda_i \gamma_i x_i(t)}{\epsilon_{2i}}. \tag{21}$$

The co-state (adjoint) variable for this particular case is defined as

$$\lambda_i(t) = \pi_i e^{\int_t^T n(\tau)d\tau} \int_t^T e^{\int_t^T n(s)ds} dt. \tag{22}$$

where $n(\tau) = \rho_i + r_i u_{1i} + \alpha_i - \gamma_i u_{2i}$.

The above-mentioned case is solved theoretically by applying the Maximum Principle to arrive at an optimal control model. The nonlinear nature of the model results in a complex analytical expression. The optimal advertising effectiveness rates obtained in Eq. (21) suggest that as the market approaches saturation, both types of advertising efforts gradually decline over time; that is, the most effective strategy is to minimize advertising efforts toward the end of the planning period. A numerical example is used to illustrate the applicability of the proposed optimal control problem.

4. Numerical illustration

In this section, we provide a numerical example that incorporates the market size over the planning period of twelve months to demonstrate the solution procedure and theoretical policies. We focus on a firm seeking to determine the best advertising strategies for its consumer durable product. The firm implements advertising efforts for both existing and new customers in order to increase its sales in a segmented market environment. We will consider two segments of the market for the numerical illustration of our model. First, we proceed with the estimation of model parameters in the following subsection.

4.1. Data analysis and model validation

In this subsection, we use real datasets to estimate the parameters, and for this estimation, three diverse data sets are collected, that is, net sales data (in Rs. crores) of company Blue Star Ltd. for Dataset-I, mobile-cellular subscription data of India for Dataset-II, and net sales data (in Rs. crores) of Britannia Industries Ltd. for Dataset-III. The Dataset-I used for this estimation spans the period from March 2007 to March 2021, as presented in Table 2, and has been sourced from https://www.moneycontrol.com/ [37]. Dataset-II, consisting of total mobile-cellular subscriptions for India from 2005 to 2023, has been sourced from the International Telecommunication Union (ITU) database (https://data.worldbank.org/indicator/IT.CEL.SETS). Dataset-III, containing annual standalone revenue from operations for Britannia Industries Ltd. for the period 2012–2025, has been sourced from the company's official annual reports available at https://www.britannia.co.in/investors/financial-performance/annual-report. To estimate, we use Ordinary Least Squares (OLS) to calibrate our model by optimizing the cost function, which is given by $\sum_i (Data_i - y_i)^2$, where $y_i = x_i$. This cost function quantifies the sum of squared differences between the observed data ($Data_i$) and the model's estimated values (y_i). Taking the initial proportion of sales to be $x_i(0) = 0.1,\ 0.07,$ and 0.2, respectively for Datasets I, II and III, and estimating the values of parameters r_i, α_i, and γ_i, considering u_{11}, u_{12}, u_{21} and u_{22} as constant in Eq. (5), we proceed with the parameter estimation and model fitting using the datasets in Table 2. The estimated parameter values are given in Table 3.

Table 3
Comparison metrics and parameter estimates.

Model under comparison	Parameter estimates	Dataset-I	Dataset-II	Dataset-III
V-W (1957) [30] model	r	0.1969	0.3559 0.2571	
	δ	4.4004e−08	2.53E−08	3.14E−07
	R^2	0.8643	0.9429	0.8903
	MSE	0.0121	0.0056	0.0056
	Variation	0.0129	0.0059	0.0059
	RMSPE	0.1138	0.0767	0.0775
Proposed model (5)	r	0.0857	0.2997	0.1274
	α	0.0060	0.0031	0.0004
	γ	0.2171	0.0436	0.1191
	R^2	0.9991	0.9612	0.9712
	MSE	7.7405E−05	0.0038	0.0015
	Variation	8.0438E−05	0.0040	0.0014
	RMSPE	0.0091	0.0630	0.0395

4.1.1. Comparison metrics

To estimate quantitatively the alignment of the simulated rate of sales and the provided real data sets of Table 2, we calculate mean square error (MSE), coefficient of multiple determination (R^2), variation, and root mean square prediction error (RMSPE). These measurements offer insights regarding the consistency of observed patterns with the model-projected sales behavior while assessing the goodness of fit of the proposed model. The comparison metrics are computed as follows:

Mean Squared Error (MSE): This metric measures the average squared differences between observed values and estimated values. It is given by:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(Data_i - y_i)^2$$

A lower MSE value indicates a better model fit.

Coefficient of Determination (R^2): This metric quantifies how well the model explains the variability of the data. It is defined as:

$$R^2 = 1 - \frac{\text{Residual SS}}{\text{Corrected SS}}$$

The value of R^2 lies between 0 and 1, where higher values indicate a better model fit.

Variation: This metric is calculated as:

$$\text{Variation} = \frac{\sum(PE_i - \text{BIAS})^2}{N-1}$$

where

$$\text{BIAS} = \frac{\sum PE_i}{N}, \quad PE_i \text{ (Prediction Error)} = OE_i - y_i$$

Here, OE_i denotes the observed values and y_i gives the model's predicted values. A lower variation value indicates a better fit.

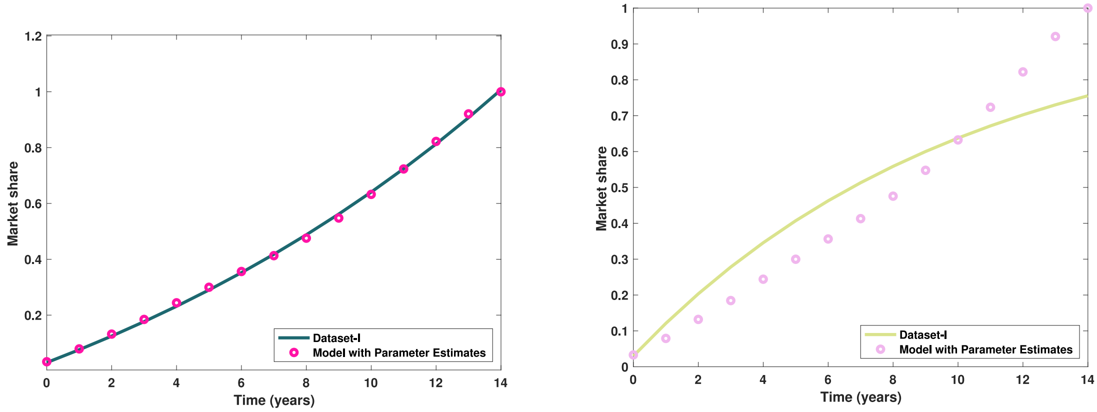
Root Mean Squared Prediction Error (RMSPE): This metric combines bias and variation to assess predictive accuracy:

$$RMSPE = \sqrt{\text{BIAS}^2 + \text{Variation}}$$

A lower RMSPE value implies better predictive performance.

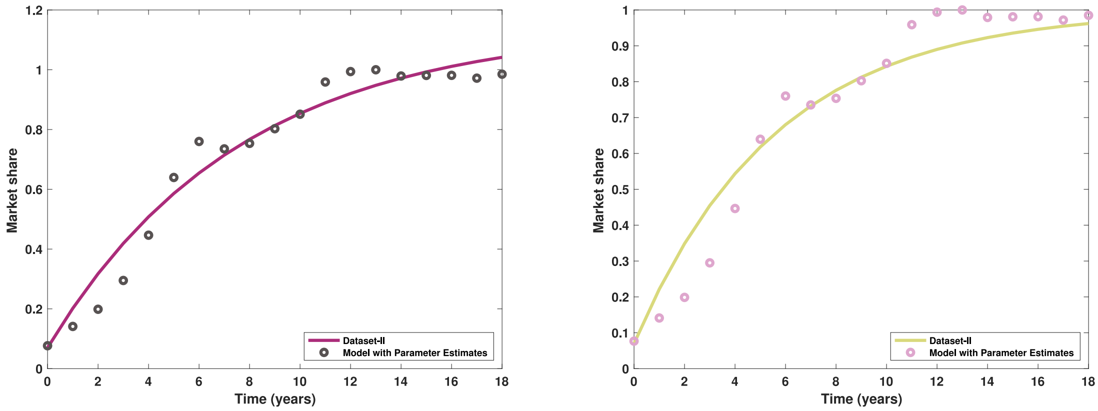
4.1.2. Data description and estimation results

From the values of the metrics computed and shown in Table 3, we see that the value of R^2 obtained for the proposed model are 0.9991, 0.9612, and 0.9712, respectively, for datasets I, II, and III, which is higher than that of the original Vidale–Wolfe model. This high value indicates that the model explains nearly all the variations in the real data, thus reflecting the fact that the model has a very strong predictive capability as it captures 99.9%, 96.12%, and 97.12% of the variations in the provided datasets. This result signifies a highly accurate fit, suggesting that the proposed model, with the estimated parameters, effectively captures the relationship between the variables while leaving minimal unexplained variance. Other measures, such as Mean Squared Error (MSE) and Root Mean Squared Prediction Error (RMSPE), further explain the validity of the model and reliability in predicting the trends seen in the real data as compared to the existing Vidale and Wolfe (1957) [30] model. These findings validate the efficacy and stability of the developed model in capturing the underlying data dynamics. Figs. 1(a), 1(b), 2(a), 2(b), 3(a), and 3(b) represents the goodness-of-fit curves, which illustrates how well the proposed model's predicted values, obtained using the estimated parameters, align with the actual real-world data.



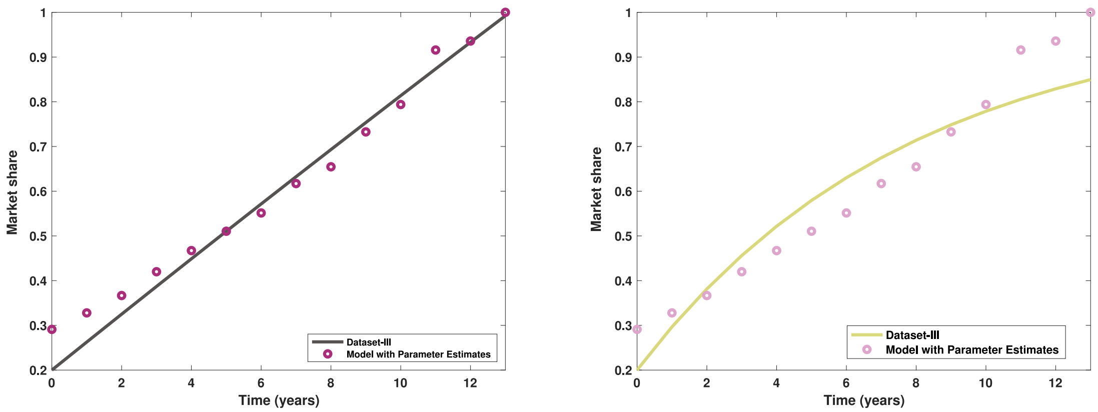
(a) (b)

Fig. 1. Goodness of fit curve (Dataset-I) for (a). Proposed model and (b). Vidale–Wolfe model.



(a) (b)

Fig. 2. Goodness of Fit Curve (Dataset-II) for (a). Proposed Model and (b). Vidale–Wolfe Model.



(a) (b)

Fig. 3. Goodness of Fit Curve (Dataset-III) for (a). Proposed Model and (b). Vidale–Wolfe Model.

Table 4
Parameter values for both segments.

Parameter	x_1	x_2	Reference
ϵ_{1i}	75	75	Assumed
ϵ_{2i}	80	80	Assumed
r_i	0.0857	0.0847	Estimated
π_i	45	48	Assumed
α_i	0.0060	0.0050	Estimated
γ_i	0.2171	0.2071	Estimated
ρ_i	0.01	0.01	Assumed

Table 5
Optimal market share.

	x_1	x_2
T_1	0.0000	0.0000
T_2	0.0466	0.0485
T_3	0.0874	0.0910
T_4	0.1234	0.1284
T_5	0.1550	0.1612
T_6	0.1826	0.1899
T_7	0.2061	0.2143
T_8	0.2255	0.2345
T_9	0.2405	0.2502
T_{10}	0.2509	0.2613
T_{11}	0.2566	0.2674
T_{12}	0.2574	0.2685

Table 6
Optimal value of advertising effectiveness rates for new customers.

	$u_{11}(t)$	$u_{12}(t)$
T_1	0.5345	0.5638
T_2	0.4851	0.5112
T_3	0.4319	0.4545
T_4	0.3770	0.3962
T_5	0.3224	0.3382
T_6	0.2693	0.2821
T_7	0.2188	0.2288
T_8	0.1711	0.1786
T_9	0.1261	0.1314
T_{10}	0.0831	0.0865
T_{11}	0.0414	0.0431
T_{12}	0.0000	0.0000

4.2. Optimal control theory simulation

The values of the parameters used in the numerical solution are provided in Table 4 in order to validate the model and determine the optimum advertising effectiveness rates. For this simulation, we use the estimated key parameters from Dataset-I (Bluestar Ltd.). Matlab is used to code and solve the optimal control problem with dynamic market size. We have considered some base values for the proposed model to solve by Matlab with initial sales taken as $x_i = 0$. The time horizon has been divided into 12 equal time periods. The number of market segments is two (i.e., M = 2), and an advertising budget needs to be allotted to maintain the existing customers and attain new customers in both of the market segments. The mentioned problem is solved by taking advertising cost as a quadratic function of the effort rate. The numerical solutions for optimal market share and advertising effectiveness rates for existing and new customers are illustrated in Tables 5, 6, and 7, respectively.

The optimal market share from each segment of the potential market are shown in Fig. 4. Fig. 5(a) and Fig. 5(b) represent the optimal advertising effectiveness rates for both new and existing customers, respectively, in each segment. Fig. 4 indicates that the relative sales begin from an initial level and thereafter increase with time. Fig. 5(a) indicated that the optimal advertising effectiveness rate for new customers was maximum in the beginning for both the segments. It can also be concluded from the same figure that the optimal advertising effectiveness rate decreases as the time period increases, which allows the decision-makers to reduce their advertising efforts at the end of the planning period. Fig. 5(b) indicated the optimal advertising effectiveness rate for new customers for both the segments, from which we can conclude that the advertising efforts were maximum in the beginning, which kept on decreasing with the passing of time.

Table 7
Optimal value of advertising effectiveness rates for existing customers.

	$u_{21}(t)$	$u_{22}(t)$
T_1	0.0000	0.0000
T_2	0.0563	0.0598
T_3	0.0983	0.1043
T_4	0.1260	0.1338
T_5	0.1405	0.1490
T_6	0.1429	0.1516
T_7	0.1349	0.1431
T_8	0.1183	0.1254
T_9	0.0948	0.1005
T_{10}	0.0661	0.0701
T_{11}	0.0340	0.0360
T_{12}	0.0000	0.0000

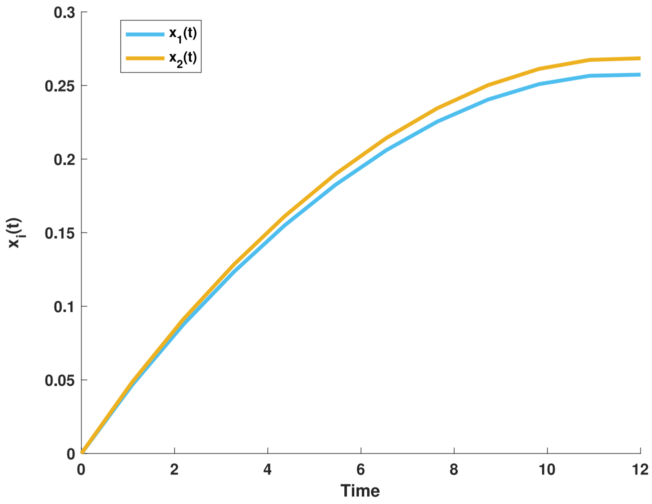


Fig. 4. Optimal market share.

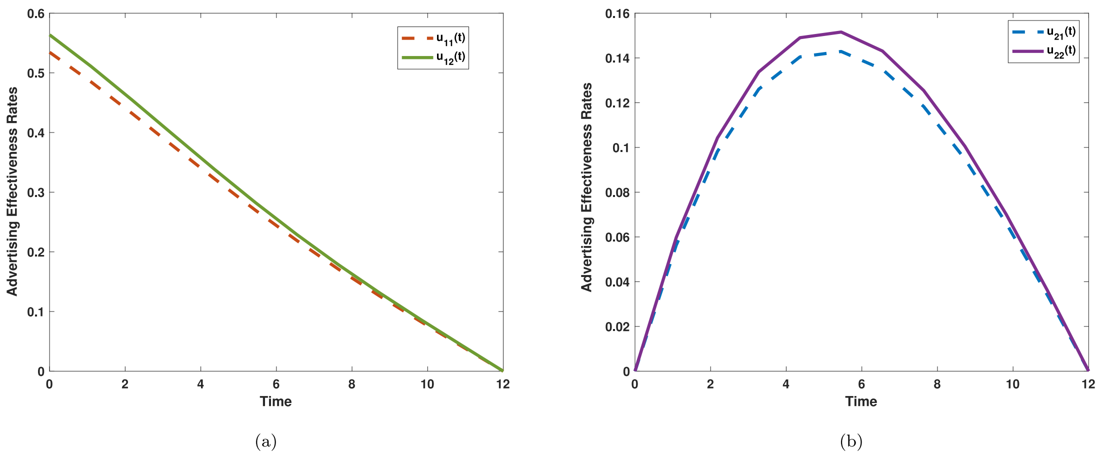


Fig. 5. Optimal advertising allocation for (a). new customers, and (b). existing customers.

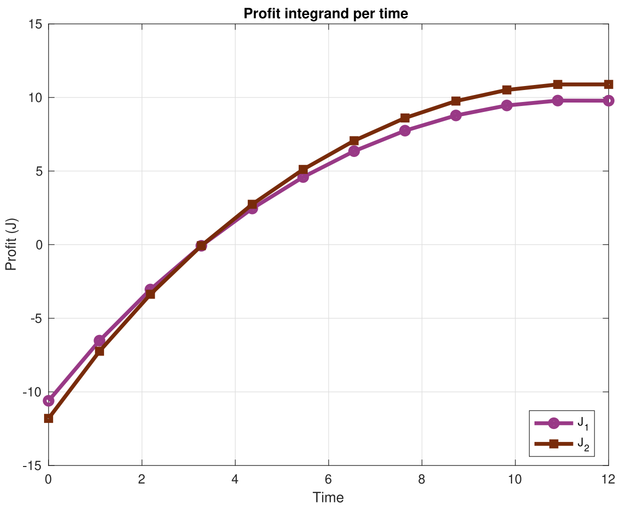


Fig. 6. Profit evolution over time.

4.3. Sensitivity analysis

In order to assess how changes to some key parameters might impact the optimal profit outcomes, a local sensitivity analysis is carried out. The performance metric in this case is the discounted profit functional.

$$J(u_1, u_2) = \int_0^T e^{-\rho t}\left(\pi x(t) - \frac{\epsilon_1}{2}u_1^2(t) - \frac{\epsilon_2}{2}u_2^2(t)\right) dt,$$

The normalized sensitivity index of J with respect to a parameter p is defined as

$$\varphi_p = \frac{\partial J}{\partial p} \cdot \frac{p}{J}, \tag{23}$$

so that φ_p represents the percentage change in profit caused by a one-percent change in p. Here p denotes parameters of interest that is, r_1, r_2, α_1, α_2, γ_1, γ_2, ϵ_{1i}, and ϵ_{2i}.

The total profit is $J = 90.17$, with $J_1 = 42.67$ (segment 1) and $J_2 = 47.50$ (segment 2) (See Fig. 6 for profit trajectory over time). The sensitivity indices thus obtained are:

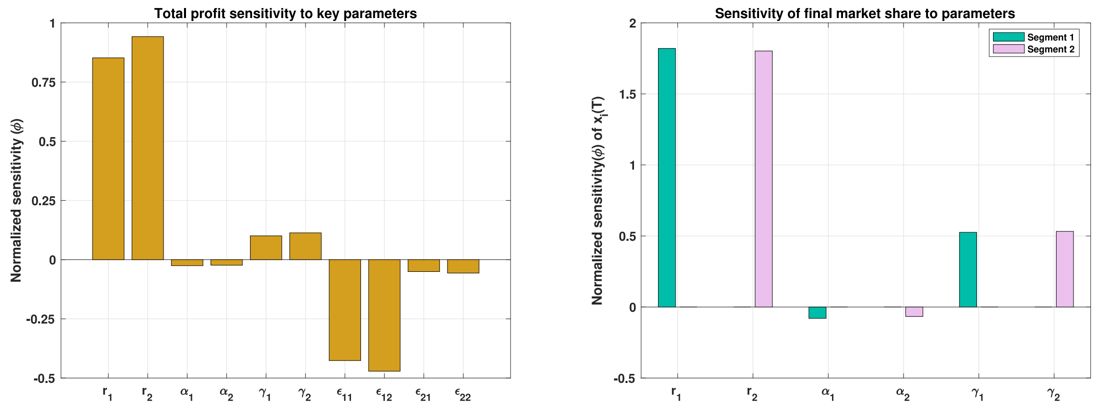
- r_1 and r_2(responsiveness of the market to advertising of new customers): Both coefficients have positive indices ($\varphi_{r_1} = 0.852$, $\varphi_{r_2} = 0.942$), which means that increased responsiveness contributes to a rise in profit. The impact is segment-oriented: r_1 increases segment 1 profit, and r_2 influences segment 2.
- α_1 and α_2 (customer natural decay rate): Both parameters have small but negative sensitivities ($\varphi_{\alpha_1} = -0.025$, $\varphi_{\alpha_2} = -0.024$), indicating that increased decay rates decrease profit through decreased retained customer numbers.
- γ_1 and γ_2 (advertising effectiveness for retaining customers): The indices are moderate but positive ($\varphi_{\gamma_1} = 0.101$, $\varphi_{\gamma_2} = 0.113$), indicating that investments in retention advertising enhance profit.
- ϵ_{11} and ϵ_{12} (cost-sensitivity for advertising to new customers) and ϵ_{21} and ϵ_{22} (cost-sensitivity for advertising to current customers): These parameters exhibit noticeably negative sensitivity indices ($\varphi_{\epsilon_{1,1}} = -0.426$, $\varphi_{\epsilon_{1,2}} = -0.471$), ($\varphi_{\epsilon_{2,1}} = -0.050$, $\varphi_{\epsilon_{2,2}} = -0.057$), which means that when advertising becomes more expensive, total profit declines significantly. We observe that the cost parameters for new customer acquisition ($\epsilon_{1,i}$) have a much stronger negative effect on profitability than retention-related costs ($\epsilon_{2,i}$) highlighting the importance of managing acquisition expense efficiently.

Overall, the findings point out that profits are most responsive to r_1 and r_2 i.e. the ease with which advertising to new customers converts the unsold market. The effectiveness in retention (γ) has a lesser but positive impact, while the natural decay rates (α) play the role of profit-reducing factors. Fig. 7(a) shows a bar plot of the profit sensitivity to key parameters.

Also, we have also evaluated the sensitivity of the final market shares $x_i(T)$ with respect to the same set of parameters, that is,

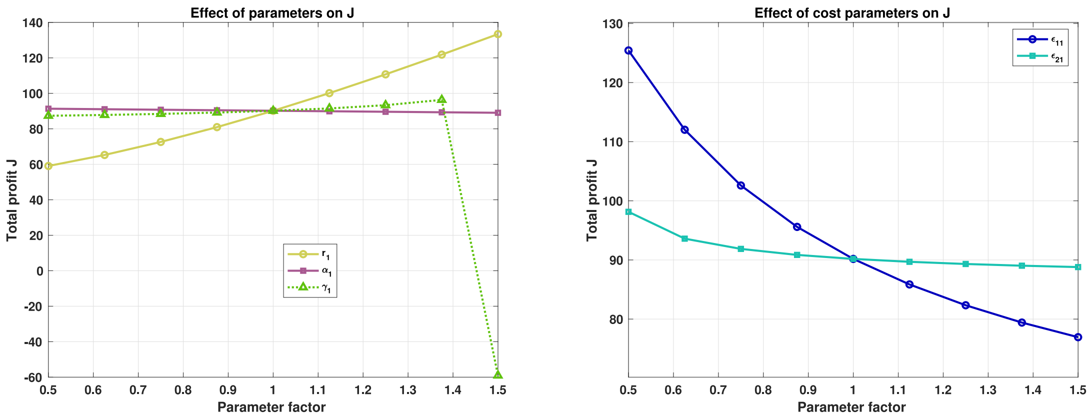
$$\varphi_p = \frac{x_i(T)}{\partial p} \cdot \frac{p}{x_i(T)}, \tag{24}$$

The simulation results show that r_i and γ_i positively influence final adoption, whereas α_i negatively affects market share, as higher natural decay causes faster customer churn. This further supports the managerial consideration that customer attrition control and the advertising responsiveness enhancement are critical levers to sustain market penetration in both segments. See Fig. 7(b) for graphical representation.



(a) (b)

Fig. 7. Sensitivity bar plots.



(a) (b)

Fig. 8. Impact of proportional changes in key parameters on the total profit J.

In addition to the aforementioned local indices, we conducted parameter sweep experiments (see Fig. 8(a)) wherein each key parameter was varied between 50% and 150% of its baseline value. The resulting profit–parameter curves demonstrate a monotonically increasing trend with respect to r_i, a decreasing trend for α_i, and illustrating nonlinear and diminishing-return behavior for γ_1 at high values which further explains that excessive retention effort becomes wasteful at some point once most customers are already retained. Furthermore, higher values of ϵ_{1i} and ϵ_{2i} significantly diminish the attainable profit (see Fig. 8(b)).

In the absence of advertising for existing customers (u_{2i}), we observe that profit is lesser in both segments as compared to when both type of advertising is implemented. The total profit J in this case is 82.6923, with Segment 1 Profit = 39.1498 and Segment 2 Profit = 43.5425. Thus, it is evident that incorporating additional advertising directed towards existing customers enhances the overall profit and improves the performance of the proposed model.

5. Managerial implications

The managerial significance of the proposed dual-control advertising framework aligns with the practices observed in leading global corporations. Coca Cola's recent advertising campaign "Share a Coke", focuses on customer engagement and builds loyalty with customers. Apple has developed the advertising campaign "Shot on iPhone"; this approach focuses on storytelling/ product to reach new customers. The results of this study quantify the significant benefit of these two types of advertising efforts through the optimal distribution of resources in each to maximize profitability, minimize customer turnover and create long-term stability for companies in the marketplace by targeting various customer segments. The findings of this research have immediate applicability to firms constructing advertising strategies in segmented markets. The results highlight that whereas early advertising initiatives of new customer acquisition are important to speed up adoption, persistent yet declining efforts are more successful in the long term as natural diffusion gains momentum. In practical terms, this is because higher responsiveness rates (r_i) of new customers lead to strong profit gains, as confirmed in the sensitivity results, meaning that aggressive initial acquisition in high-response segments rapidly expands adoption. Hence, firms should concentrate acquisition advertising in segments where conversion propensity is highest to maximize market penetration at lower cost.

Concurrently, focused campaigns towards existing customers drastically reduce attrition, helping preserve the firm's market share against competitive forces. This managerial outcome is driven by the decay-reduction effect of advertising towards existing customers captured by $\delta_i(u_{2i}) = \alpha_i - \gamma_i u_{2i}$. Our parameter sweep experiments further reveal that while higher γ_i (retention effectiveness) improves profits, excessive retention spending eventually yields diminishing returns when almost all customers are already retained. Thus, a moderate but sustained strategy is most cost-effective for customer retention.

The segment-specific approach makes it possible for firms to identify and focus on high-response market segments with higher conversion rates through (r_i) and stronger retention benefits (γ_i), which in turn supports smarter budget planning by directing more resources toward the most impactful customer segments. For instance, when a segment exhibits lower natural decay (α_i), that segment becomes more profitable to defend via targeted retention campaigns.

From a managerial perspective, it is suggested that budget allocation need not be uniform over time or across segments. Instead, it is beneficial for firms to prioritize advertising efforts directed towards acquiring new customers while also planning moderate but consistent advertising efforts towards retaining the already existing customers. The cost-sensitivity parameters ϵ_{1i} and ϵ_{2i} indicate that acquiring more customers become more expensive as markets become more saturated, thereby making early investments more financially advantageous. Sensitivity analysis indicates that an increase in ϵ_{1i} significantly reduces profitability, highlighting the necessity of identifying more cost-effective strategies for attracting new customers.

The empirical findings demonstrate that the proposed model significantly surpasses the classical Vidale–Wolfe model in terms of predictive accuracy and precision. This underscores its practical applicability in enhancing advertising strategies, demand forecasting, and budget allocation to further support advertising efforts. Additionally, we can say that markets with lower decay and stronger conversion tendencies show better long-term profit outcomes, enabling firms to prioritize resources based on measurable marketing responsiveness.

By skillfully trading off these two levers, that is, acquisition and retention efforts across different market segments, firms can have both growth and stability, maximizing profitability across diverse market niches. The managerial insights obtained from our study offer explicit guidance: prioritize substantial investment in high-response acquisition at the outset, sustain ongoing targeted retention efforts, and consistently monitor parameter responsiveness to dynamically adjust spending throughout the planning period.

6. Conclusion

The advertising strategy for existing customers aims to maintain their engagement and allegiance to a product, ensuring they remain connected and less likely to drift towards competing brands. Conversely, advertising strategies directed towards new customers employ a distinct approach, emphasizing comprehensive information dissemination about the product's core features and advantages. This approach seeks to attract and persuade potential new customers to adopt the product in different segments. We have used the concept of market segmentation in Vidale–Wolfe model to decrease the decay rate of existing customers and study the optimal advertising effectiveness rate of both new and existing customers towards a segmented market. As advertising campaigns directed towards existing customers play an important role in reducing the decay constant, implementing this is crucial in customer retention and slowing adopter loss. While advertising directed to new customers needs to be a priority for maximum advantage, strategic resource balancing for retaining existing customers can enhance long-term market stability and efficiency in resource use. This highlights the importance of targeted implementation of control to boost efficiency in resource consumption to attain the maximum possible number of customers. It can be concluded from the simulation that the optimal advertising effectiveness rate decreases as the time period increases, which allows the decision-maker to reduce their advertising efforts at the end of the planning period. The advertising efforts were maximum at the beginning of the planning period, which kept on decreasing with the passing of time. Thus, it is recommended that for the most effective control strategy, firms implement heavy advertising efforts to give momentum to the diffusion process for adopters initially, but as the word-of-mouth process gains momentum, the demand for advertising efforts diminishes. Therefore, it is advisable that the decision maker lower the advertising effort at the end of the planning period. To validate our model, estimation of parameters was done using real datasets to estimate key parameters and then compared with the original Vidale–Wolfe model. Some comparison measures were utilized to assess the validity of the model and demonstrate that the model is appropriately fitted to the data. The results demonstrate the efficacy of the model in characterizing the system dynamics and validate its use in real life. Sensitivity analysis further showed that responsiveness parameters to acquiring new customers (r_1, r_2) have the strongest positive effect on profit, and effectiveness of retention (γ_1, γ_2) has a moderate positive effect. Natural customer attrition rates (α_1, α_2) negatively impact profitability. In addition, the cost-sensitivity parameters $(\epsilon_{1i}, \epsilon_{2i})$ indicate that higher acquisition and retention costs significantly reduce attainable profit, thereby emphasizing the importance of cost-efficient advertising strategies. Together, these results shed light on how firms can effectively structure advertising strategies that balance acquisition and retention to maximize long-run profit. Finally, the model offers both theoretical contribution and practical advice for decision-makers wishing to optimize advertising expenditures in segmented markets.

Limitations and future research directions

While the model embodies the fundamental trade-offs between acquisition and retention in segmented markets, it does have some limitations. The analysis employs constant responsiveness parameters and does not necessarily address explicit competitive or budgetary constraints. In addition, advertising is considered a single-channel activity when in reality firms use multiple platforms with heterogeneous impacts. Another limitation is that customers within a segment are assumed to behave similarly, but in reality they differ in loyalty, price sensitivity, and likelihood of switching brands. The simplifying assumptions are necessary to make the framework analytically solvable but limit its scope of application when extended to more complex settings.

Future extensions can include adding stochastic customer dynamics, which would reflect uncertainty over adoption and churn rates, making the model closer to reality for fluctuating markets. A further extension is to consider budget-constrained and competitive advertising contexts, where companies have resource constraints or competitor advertising that changes market behavior. Incorporation of cross-segment externalities, in which advertising in one segment indirectly affects adoption elsewhere—for instance, promotions targeting younger consumers spurring interest among older segments through word of mouth might also make the model richer. Just as well, adding multi-channel advertising approaches would reflect the real-world fact that companies distribute their campaigns across various media, each with varying costs and effects. Lastly, since the assumption of constant responsiveness is untenable, it is preferable to relax the same and accommodate time-varying parameters to better capture how customer sensitivity changes throughout the planning period. In combination, these extensions would enhance the model's realism and give more profound managerial insights into dynamic advertising allocation.

Funding

Authors declare that no funding was received.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data used for parameter estimation. Data described in Table and cited in manuscript.

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