ISSN No. 2454 - 1427

CDE
January 2022

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Working Paper No. 319

http://www.cdedse.org/pdf/work319.pdf

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Bargaining for Assembly

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January 17, 2022

Abstract

An assembly problem refers to a situation where a buyer wants to purchase a fixed number of complementary items from sellers holding an item each. We model complementarity using graphs where nodes represent items, and edges between two nodes represent a complementary relationship between these items. The buyer wants to purchase a feasible path in the graph, i.e., a path of desired length, where the sum of valuations of the sellers owning the items do not exceed buyer's own valuation. A seller is critical if he lies on every feasible path. We examine subgame perfect equilibria of an infinite horizon alternate-offer bargaining game between the buyer and the sellers. We show that there exist equilibria where the buyer can extract full surplus within two periods if and only if (a) there are no critical sellers and (b) there exist at least two feasible paths with minimum sum of seller valuations. We also characterize the upper bounds on buyer's surplus when she cannot extract full surplus. Thus we characterize the trade-off between complementarity and competition in terms of buyer's equilibrium surplus share in assembly problems.

JEL Classification: C78

Keywords: Assembly, Bargaining, Competition, Complementarity, Contiguity, Holdout

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†gupta.dhritiman@gmail.com, O.P. Jindal Global University, Sonipat, India. We are grateful to Professors Prabal Roy Chowdhury and Arunava Sen for their guidance and encouragement. We are also grateful to Steven Williams, Jun Xiao, Huseyin Yildirim, participants of the Delhi Economic Theory Workshop, Conference of Economic Design, York, 2017, Asia Pacific Industrial Organization Conference, Auckland, 2017, and Economics Seminar, Shiv Nadar University, Delhi, 2017. Special thanks to Albin Erlansson for reading a draft of this paper and giving many useful suggestions. Travel grants from Royal Economic Society and TERI School of Advanced Studies are gracefully acknowledged.

1 Introduction

In many real life situations a buyer needs to acquire multiple items to implement a project. Examples include acquiring multiple plots of land to set up a factory, hiring faculty to set up an academic department, acquiring different molecules to make a new drug, among others. In most of these situations the items are owned by different individuals. Consequently, the buyer needs to bargain successfully with multiple sellers owning distinct sets of items. We refer to such situations as assembly problems.

Existing literature on assembly problems (see Section 3) largely assumes perfect complementarity of items, i.e., the buyer must acquire all items available to implement her project. It focuses on the effect of different bargaining protocols on the incidence of strategic delays or holdout. We want to analyze whether holdout persists when we relax the assumption of perfect complementarity. When a greater number of items are available than required, the buyer may be able to substitute an item for another, which is likely to reduce the ability of the sellers to engage in strategic delays. However, the buyer's ability to substitute between items may be restricted, leading to a range of cases between perfect substitutability and perfect complementarity.

We model the assembly problem using graphs. This approach was introduced by Sarkar (2017). Each node on a graph represents an item. A pair of nodes is connected by an edge if the buyer can use them together. A sequence of connected nodes is called a path. The buyer wants to purchase a feasible path, i.e., a path of desired length or number of nodes, where the sum of valuations of the sellers owning the items does not exceed buyer's own valuation. In other words, a feasible path must contain the minimal number of items required to implement the project, and the surplus, i.e., the excess of buyer's valuation over that of sellers on this path must be positive.

If a graph contains only one feasible path, the corresponding items are perfectly complementary — the buyer can only realize a positive valuation by acquiring all items in this path. Items can be substitutes when there are more than one feasible path in the graph and the buyer can purchase any of the collections of items represented by these paths. Items can also exhibit complementarity when feasible paths have an intersection— the buyer must acquire items in the intersection to realize positive valuation. Sellers corresponding to items showing complementarity in this sense are called critical sellers in our model. Examples of such assembly problems is provided in Section 2.

We further model the strategic interaction between the buyer and the sellers as a complete information infinite horizon bargaining problem. In such models, the agents bargain over the price at which trade may take place, and it determines the share of the surplus that accrues to the agents as net payoffs. We show that full surplus extraction by the buyer within two periods is a subgame perfect equilibrium of our bargaining model if and only if (a) players are sufficiently patient, (b) there are no critical sellers and (c) there exist at least two feasible paths with minimum sum of seller valuations. We also characterize the upper bounds on buyer's surplus when she cannot extract full surplus. Condition (b) highlights the bearing of complementarity on surplus extraction, while (c) also highlights the bearing of asymmetric valuations.

Our bargaining protocol is a natural extension of the protocol used by Rubinstein (1982) to multi-agent situations which has also been used by Roy Chowdhury and Sengupta (2012). Further, we focus on non-cooperative behavior among sellers like in most of the literature on holdout. Hence our contribution is twofold: one, to characterize the possibility of holdout in multi-agent non-cooperative bargaining situations modelled using graphs; second, unlike the literature, we bring out the role of asymmetry in seller valuations in holdout situations.

Our paper is structured as follows. We motivate the relationship between graphs and the assembly problem in Section 2. We discuss the relevant literature subsequently in Section 3. We lay down the preliminaries of our model and present two important results from the literature in Section 4. Then we present our main results for different cases of our model in Section 5. Finally, Section 6 offers discussion of the main results. All proofs are presented in the Appendix.

2 Graphs and assembly

A graph is a collection of two sets: one is a set of objects called "nodes" and the other is a subset of all possible pairs of such nodes, called "edges". These can be visualized as circles representing nodes and lines connecting a pair of circles respectively. If the pairs are ordered, the graph is called directed and the corresponding edges are represented with arrows. A path in a graph is a sequence of nodes connected by edges. The length of a path refers to the number of nodes it contains. See Bollobás (1998) for a comprehensive treatment of the theory of graphs. We will be using undirected graphs in our problem.

Suppose the government wants to build a flyover by combining three ($k = 3$) plots out of 6 available plots ($n = 6$) numbered 1 through 6. For simplicity, let us temporarily assume that all sellers have identical valuations. Not all combinations of three plots are feasible because some pair of plots may have protected forest cover. Each plot is represented by a node on a graph; plots that do not have any forest cover between them are connected by an edge between corresponding nodes on the graph. The government then needs to pick a path of length 3 on the resulting graph. Let us now interpret alternative graph structures in this context.

Consider Figure 1. There is no forest cover between plot 1 and any other plot, but the path between every other pair of plots have forest cover. Plot 1 is perfectly complementary to other plots in any assembly plan to construct the bridge involving three plots. We refer to such a plot as a critical plot and the corresponding seller as a critical seller. Note that there can be more than one critical seller (Figure 2).

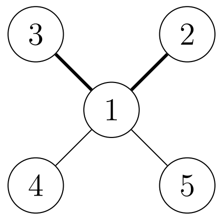

Figure 1: A feasible path in the star graph when $k = 3$; seller 1 is critical.

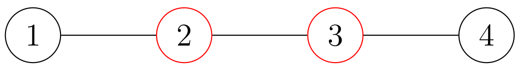

Figure 2: A line graph with critical sellers 2 and 3;k=3

Consider any pair of plots in Figure 3: either there is no forest cover between them, or there is uninterrupted access from one plot to other via some other plot. This case represents perfect substitutability between any pair of nodes: if a particular plot in a combination is replaced with some other plot, the resulting combination remains feasible.

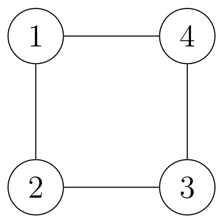

Figure 3: A cycle of length 4.

Consider plots 1, 2 and 3 in Figure 4: there is no forest cover between 1 and 2 or 2 and 3, and consequently, there is an uninterrupted access between 1 and 3 via 2. Similarly for the set consisting plots 4, 5 and 6. The government can substitute path 456 with path 123 but cannot substitute a node within a path with a node outside a path. In this sense, nodes on a particular path are perfect complements but the paths are perfect substitutes.

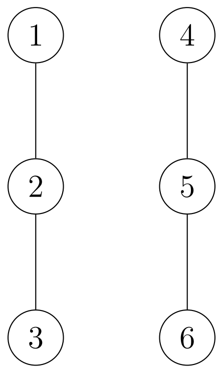

Figure 4: Graph with disjoint feasible paths; k=3

Consider Figure 5 containing a graph we call *oddball*. There is no critical seller, no pair of paths is disjoint and not every node on a feasible path can be replaced by another node. There is a cycle but it is unlike Figure 3. This is a case where a plot on a path can be substituted by a set of plots to maintain feasibility. For instance, consider the path 123 — 1 or 3 can be replaced by 4 but 2 can be replaced by 4 and 5, while plot 3 is going to be wasted.

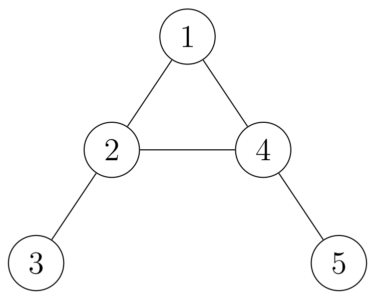

Figure 5: An oddball graph, $n = 5$, $k = 3$

The idea of congruence has natural appeal in land assembly because the buyer needs rights of passage from one plot to the other to implement the project. Another natural example is that of a mobile service provider trying to purchase spectrum in multiple districts. The buyer values contiguity of districts where spectrum is acquired because it ensures seamless connectivity across the coverage area. A rather unconventional example in this context is a situation where a music composer wants to assemble different parts from other songs to compile a new score. But these songs are owned by different copyright holders. Portions of songs have to be harmonically close enough to each other to be combined to yield a meaningful score.

Certain assembly problems may use a different idea of congruence that may not immediately be amenable to our graph-theoretic treatment. Consider a pharmaceutical company which wants to create a drug by assembling molecules owned by different companies. Let us use Figure 1 to illustrate this case, renaming plots as molecules. Let us say that the only combinations of molecules that work are 1, 2 and 3 or 1, 4 and 5. But now 2, 1 and 4 may not always make a feasible combination because of their chemical properties. In a similar vein, suppose 1 represents a factory, 2 and 4 are wholesalers while 3 and 5 are retailers; a brand wants to create a complete supply chain by signing agreements with the factory, a wholesaler and a retailer. Notice that now 214 no longer remains feasible as it contains two wholesalers and no retailer. In this case, it makes more sense to use graphs like Figure 4, repeating 1 in both paths, or keep 1 out of the situation completely and consider disjoint paths with a retailer and a wholesaler each.

3 Literature

In situations where a buyer needs to buy complementary inputs from different sellers, hold out is expected to cause inefficiencies, e.g., strategic delays, implementation of sub-optimal projects (e.g., purchase of a set of objects that does not correspond to the highest surplus) or complete breakdown of negotiations. This problem has been studied in the land assembly context (Asami, 1985; O'Flaherty, 1994; Cai, 2000, 2003; Menezes and Pitchford, 2004; Miceli and Segerson, 2012; Roy Chowdhury and Sengupta, 2012; Göller and Hewer, 2015; Xiao, 2018).

Roy Chowdhury and Sengupta (2012) have studied the problem of a buyer and multiple sellers bargaining with one another, where all items are complementary. They focus on the role of transparent protocols (i.e., where offers and counteroffers are fully observed by agents) and outside options: buyer can extract higher surplus with transparent protocols if she has an outside option; holdout may be unavoidable with less transparent protocols even in presence of an outside option. We use the same transparent protocol they propose. The buyer has no outside option, similar to their benchmark model. We introduce competition among sellers in the model by allowing for more sellers than the number of items required. Our graph theoretic approach allows us to explore different degrees of complementarity among items and relates it to the phenomenon of holdout in an intuitive way.

A number of contributions in the literature have applied protocols where the buyer engages in a sequence of bilateral negotiations with sellers (Cai, 2000; Suh and Wen, 2006, 2009; Li, 2010). Delay is embedded in such protocols in the sense that at least k periods are required for successful assembly if the buyer needs to assemble k units. Unless the buyer's budget per period is limited, or the application in question involves bargaining with agents in different levels of supply chain (e.g., wholesaler, retailer etc.), a rational buyer would minimize such delay.

Our contribution is distinct from the literature on bilateral trade on networks (see the survey by Manea (2016)): in this literature, bilateral trade takes place in each period between a random pair of nodes on a network. In contrast, we use a network to model the complementarity of items owned by sellers. The buyer is isolated from this network, but wants to purchase all nodes on a feasible path. She can make an offer to any seller and vice versa, but no seller can make any offer to another seller. It is also distinct from the literature on spectrum and package auctions (see the survey by Bichler and Goeree (2017)): in such auctions, multiple buyers have possibly different valuations over different "packages" of radio spectrum. In contrast, our single-minded buyer has the same valuation over every feasible path.

Sarkar (2017) investigated the existence of direct mechanisms that are "successful" in the sense of Myerson and Satterthwaite (1983)[1] when agents have private and independent valuations and seller valuations are drawn from the same prior (also see Kominers and Weyl (2012); Grossman et al. (2019)) . Although a successful direct mechanism may exist for certain priors, the form of such a mechanism may not be very intuitive. In contrast, bargaining has a natural interpretation in a complete information framework. It also enables us to study the equilibrium strategies of the agents in detail.

A natural follow-up of our exercise is to investigate the impact of formation of seller coalitions on equilibrium payoffs (see Ray (2007) for a survey of coalition formation). In our discussion section, we provide an example to show that if the sellers are allowed to form coalitions, the buyer may not be able to extract full surplus even when sellers have identical valuations.

4 Preliminaries

Sellers of items are located on nodes of a graph. Two sellers are connected by an edge if the corresponding items are complementary in buyer's assembly problem. A path is a sequence of connected nodes. The buyer wants to purchase a path of desired length[2], say k. This implies that the buyer can combine any k complementary items on the path to

[1] A mechanism is "successful" in this sense if it is ex-post efficient, interim incentive compatible, interim individually rational and ex post budget balanced.

[2] This can be relaxed to include any special graph of a fixed size. Rights of passage directly motivates the desire to purchase a path in our case.

produce output. We denote a path by $\mathcal{P}$ and the corresponding sum of seller valuations by $\mathcal{S}$. An assembly problem with complete information is a tuple: $\langle \Gamma, k, v, \delta \rangle$. Here Γ is a graph of order n; positive integer k is the desired minimum length of the path buyer is interested in purchasing; if the buyer cannot acquire a path of size k or more, the project is not feasible and the value he gets is normalized to 0; the first component of $v \equiv (v_0, v_1, \ldots, v_n)$ denotes the valuation of the buyer for a path of length k or more, and other components denote the valuation of the sellers for their respective items; the real number $\delta \in [0, 1]$ denotes the common rate at which the $n+1$ agents discount future payoffs. We assume that there exists a path $\mathcal{P} \in \Gamma$, such that it results in a positive surplus: $v_0 - \sum_{i \in \mathcal{P}} v_i \geq 0$. Each such path would be referred to as a feasible path. Given such a graph Γ, the expression $\max_{\mathcal{P} \in \Gamma} \left(v_0 - \sum_{i \in \mathcal{P}} v_i \right)$ is referred to as "full surplus" or "efficient surplus".

The variety of possible graph structures can be large since a graph with n nodes can have up to $\binom{n}{2}$ edges. For a given k, we can categorize single component graphs into four mutually exclusive and exhaustive classes as follows. In general, a graph may have multiple components from different classes.

Graphs with Critical Sellers: A seller is critical if he lies on every feasible path (see Figure 1 and Figure 2). Notice that if there is only one feasible path in Γ, all sellers in that path are critical. But if there are multiple feasible paths, a seller must lie in their intersection in order to qualify as critical. In other words, the closer k is to n, the higher is the number of critical sellers. In the class of graphs with critical sellers, referred to as Γ^*, items belonging to critical sellers are not substitutable but those belonging to non-critical sellers are substitutable in a limited sense.

Single component graphs without a critical seller can be of following kinds.

Graphs with a $k+1$ Cycle: Graphs with cycles of order $k+1$ are referred to as $\Gamma^{\triangle}$ (see Figure 3). Here, every item on a feasible path can be substituted by another item on the graph. Note that when Γ is a complete graph of order n, which we denote as an assembly problem by $\langle n, k, v, \delta \rangle$, it belongs the the class $\Gamma^{\triangle}$.

Graphs with Disjoint Paths: Graphs with only disjoint feasible paths are referred to as Γ^D (see Figure 4). Here, no individual item is completely substitutable, but a feasible path can be substituted by another feasible path.

Oddball Graphs: Finally, the class of graphs where (i) there is no cycle of length $k+1$, (ii) no two feasible paths are disjoint and (iii) the intersection of all feasible paths is empty, is referred to as Γ^O (see Figure 5). We will call this class *oddball*. In this class, items in the intersection of two or more feasible paths cannot be substituted with items on these feasible paths, but they can be substituted with items on other feasible paths.

4.1 Bargaining protocol

We apply a variant of the Rubinstein bargaining protocol which is due to Roy Chowdhury and Sengupta (2012). In each period, either the active sellers (i.e., the sellers who have not completed trade and left the market) propose individual price offers (or equivalently, surplus shares) to the buyer or the buyer proposes a vector of price offers (or surplus shares) to active sellers. Suppose the buyer proposes first. The sellers can individually accept or reject the offer. The sellers who reject buyer's offer propose individually to the buyer in the next period that the buyer may accept or reject. If the buyer reaches an agreement with any of the sellers in some period, she immediately purchases his plot and the seller leaves the market. The buyer and the remaining seller then resumes bargaining. The game continues till the buyer is able to purchase at least one feasible path.

We allow the buyer to utilize negative surplus offers to exclude particular sellers from the bargaining process — such offers translate into prices that are less than seller's valuation and therefore, rejected. This facilitates the buyer avoid the commitment involved in a cash offers bargaining protocol. Also, such negative offers enable the buyer to choose sequences of sellers to bargain with in each period as discussed in our introduction. Notice that a seller cannot possibly make a negative offer to the buyer in our setting, since it delays trade with the buyer or eliminates the prospect of trade[3]. Bilateral bargaining models, like that by Rubinstein (1982) do not use this feature, while multilateral models like Roy Chowdhury and Sengupta (2012) do.

4.2 Existing results

Two special cases of bargaining for assembly in our sense are covered below for completeness. The bilateral game studied by Rubinstein (1982) is a special assembly problem with $n = k = 1$. Here the only seller present is critical. The Subgame Perfect Nash Equilibrium of this game, which is now a standard result, is presented below without a proof.

> Consider the model where the buyer bargains with one seller for one input: $\langle n = 1, k = 1, v_0 > v_1, \delta \rangle$. There is a unique SPNE of the model described as

[3] Allowing negative offers to sellers makes more sense when there are multiple buyers.

follows: agent i proposes a share $\frac{\delta}{1+\delta}$ of the surplus to j whenever she has to propose, and accept any share at least equal to $\frac{\delta}{1+\delta}$ whenever j has to propose. The game ends in the first period itself, with buyer proposing $\frac{\delta}{1+\delta}$ to the seller and the seller accepting it.

The model studied by Roy Chowdhury and Sengupta (2012) is a special assembly problem with $n = k \geq 2$ and all seller valuations are identical. Since the buyer wants all n plots, all sellers are critical here.

Consider the model $\langle n \geq 2, k = n, v_1 \leq \cdots \leq v_n, v_0 > \sum_{i=1}^{n} v_i, \delta\rangle$. The buyer's equilibrium payoff cannot be more than $\frac{1-\delta}{1+\delta}$ of the full surplus for any $\delta > 0$.

Recall that ours is a cash offers game, where sellers leave the market once trade has taken place. When non-critical sellers that buyer needs to bargain with have left the market, the continuation games involving the buyer and critical sellers will have features identical to these special cases cited above. We will be using this insight to identify bounds on buyer's equilibrium surplus share in the presence of critical sellers.

4.3 Two examples

Unlike the models covered in the previous subsection, we are interested in assembly problems where not all sellers are critical — this corresponds to the cases where Γ contains at least two feasible paths. Also, we allow for arbitrary seller valuations. The essential arguments of our main results presented in the next section are illustrated below using the simplest such cases: In Example 1, buyer bargains for one item from two sellers holding an item each and having the same valuation; In Example 2 she bargains for one item from two sellers holding an item each and having different valuations.

Example **1** Consider the model $\langle n = 2, k = 1, v_0 > v_1 = v_2, \delta\rangle$. Suppose the buyer makes offers of zero surplus to seller 1 and negative surplus to seller 2. If seller 1 rejects the buyer's offer, he would compete with seller 2 in the next period and offer the entire surplus to the buyer. If sellers 1 and 2 are making offers in the first period, they cannot make equal positive claims: one of the sellers have the incentive to reduce her claim and increase payoff. On the other hand, if their claims are unequal, the seller with the lowest claim has the incentive to increase her claim slightly and increase his payoff. Consequently, none of the sellers 1 and 2 can extract any surplus. The game ends immediately with the buyer extracting full surplus. The equilibrium outcome is identical even when the sellers are proposing first.

The situation described in Example 1 is identical to the well-known Bertrand model of price competition between firms producing the same product at identical marginal costs. Note that in our model the competition is among feasible paths. Consequently, the richness of the underlying graph structure allows for results that are richer than simple Bertrand competition.

The simple example below illustrates that the buyer may not be able to extract efficient surplus when seller valuations are not identical. This example is in the lines of Blume (2003) who characterizes a class of equilibria in the Bertrand model of price competition when firms have asymmetric marginal costs.

Example **2** Consider the model $\langle n, k, v, \delta \rangle$ such that $n = 2; k = 1, v_1 < v_2 < v_0$. Consider the following strategies of the sellers: in any continuation game where the two sellers are making offers, seller 1 offers to sell at a price of v_2 and seller 2 mixes prices in $(v_2, v_2 + \gamma)$, $\gamma > 0$, with uniform probability. In any continuation game where the buyer is making an offer, seller 1 accepts a surplus of at least $\delta(v_2 - v_1)$ and seller 2 accepts any positive surplus. Given these strategies, following is a best response for the buyer: in any continuation game where the buyer is making an offer, she offers a surplus of $\delta(v_2 - v_1)$ to seller 1 and a negative surplus to seller 2. In any continuation game where the sellers are making an offer, she accepts any surplus offer which is less than or equal to $v_2 - v_1$. If the buyer proposes first, trade takes place in the first period itself with seller 1; seller 1 is able to extract a surplus of $\delta(v_2 - v_1)$. If the sellers propose first, trade takes place in the first period, where seller 1 is able to extract a surplus of $(v_2 - v_1)$. To check that this is an equilibrium, note that when making an offer, buyer cannot offer any higher surplus to seller 1 as it would be accepted. The buyer cannot offer positive surplus to seller 2, since he would accept it. Any lower surplus offer would be rejected by seller 1. When sellers are making offers, the buyer cannot reject the offer of seller 1 either because that would reduce her share of surplus. Seller 1 cannot reduce his offer because it would be accepted. Any higher offer by seller 1 would be rejected, thus leading to a lower surplus for him. If $v_1 < v_0 < v_2$, only trade with seller 1 is feasible, making him a critical seller.

5 Results

In this section, we consider subgame perfect equilibrium outcomes of our simultaneous-offer protocol in assembly problems $\langle \Gamma, k, v, \delta \rangle$ where Γ has at least two different feasible paths and v is any arbitrary valuation profile. In the next subsection, we show that the buyer extracts full surplus within two periods if the underlying graph does not contain a critical seller and at least two feasible paths have the minimum sum of seller valuations. The following subsection characterizes the highest share of the surplus the buyer can achieve in the presence of critical sellers.

5.1 Full surplus extraction

The following result characterizes when it is possible for the buyer to extract full surplus in an equilibrium of our protocol.

Theorem **1** *There exists an equilibrium in a land assembly problem where the buyer extracts full surplus in at most two periods if and only if*

- *the discount factor δ is sufficiently large,*
- *there does not exist a critical seller in the underlying graph, and*
- *there exist at least two feasible paths with the minimum sum of valuations.*

Remark **1** The existence of the critical seller on the graph is independent of the sum of seller valuations on feasible paths. For illustration, consider Figure 1: this graph contains a critical seller and multiple feasible paths when $k = 3$.

For any assembly problem, Theorem 1 gives conditions on the graph Γ, valuation profile v and discount factor δ such that the buyer extracts full surplus within two periods. The proof of this result, given in the Appendix, breaks it down into several Propositions. Propositions 1-3 correspond to the "if part" of this result, constructing equilibria with full surplus extraction when the valuations of sellers are equal and the underlying graph does not contain a critical seller, viz., (a) graphs with $k + 1$ cycles, (b) graphs with disjoint feasible paths and (c) oddball graphs. Proposition 1 shows that in case (a), buyer can extract full surplus in the first period regardless of who is making the first offer. Propositions 2 and 3 pertain to cases (b) and (c) respectively. In these cases, buyer can extract full surplus only in the second period if she is making the first offer, provided δ is sufficiently large. In both these cases, buyer extracts full surplus in the first period if the sellers are making the first offers, regardless of δ. These results are proved by using arguments similar to the one in Example 1. These arguments essentially rely on using competition between sellers on pairs of feasible paths, and hence they are easily generalizable to the case where there are at least two feasible paths with the minimum sum of seller valuations. Proposition 4 shows that there is no equilibrium with full surplus extraction when there is a critical seller in the assembly problem. Proposition 5 shows the impossibility of full surplus extraction when there is only one feasible path corresponding to the minimum sum of valuations. This result is proved by using an argument similar to the one in Example 2.

Existing results covered in Section 4.2 provide upper bounds of buyer's surplus share in equilibria of assembly problems where all sellers are critical and valuations are identical. In contrast, Theorem 1 presents the other extreme: if no seller is critical and valuations are sufficiently identical (i.e., there exist at least two feasible paths corresponding to minimum sum of valuations), buyer extracts full surplus in an equilibrium if δ is sufficiently large. We now turn to cases that between these two extremes, where either some of the sellers are critical or valuations are not identical.

5.2 Bounds on buyer's equilibrium surplus share

The following result provides a bound on buyer's equilibrium surplus when the underlying graph does not contain a critical seller and the feasible path with the minimum sum of valuations is unique.

Theorem **2** *Consider an assembly problem without a critical seller where there is a unique feasible path with the minimum sum of valuations. Buyer's surplus cannot exceed* $v_0 - \mathcal{S}_2$ *in any equilibrium, where* $\mathcal{S}_2$ *is the second minimum sum of valuations across feasible paths.*

Example 2 is an immediate case where this bound is achieved exactly. The proof given in Appendix B uses method of contradiction. It shows that one of the sellers with whom trade takes place, would have a profitable deviation in case the surplus of the buyer exceeds this bound. Hence this case cannot be an equilibrium outcome.

Remark **2** *A variety of equilibrium outcomes are possible here: the buyer may choose to trade with sellers on the efficient path, or she may choose to trade with sellers on other inefficient paths. Further, trading may not complete in the first period itself. Regardless of such inefficiencies, buyer's surplus cannot exceed* $v_0 - \mathcal{S}_2$ *in any equilibrium.*

The next result provides a bound on buyer's equilibrium surplus share when the underlying graph contains a single critical seller. Notice that this result holds for arbitrary valuations.

Theorem 3 *Consider an assembly problem with exactly one critical seller. The buyer's share of surplus cannot exceed* $\frac{1}{1+\delta}$ *in any equilibrium.*

The bilateral trade model due to Rubinstein (1982) (Section 4.2) is an immediate example where this bound is achieved exactly. The proof given in Appendix C uses method of contradiction. We show that if such an outcome were possible, the critical seller realizing a surplus share less than $\frac{1}{1+\delta}$ has a profitable deviation. Hence this case cannot be an equilibrium outcome.

The remark made with respect to inefficiency of the possible equilibrium outcome in the previous result also applies here. Since the total surplus is maximized when the buyer trades with sellers on the efficient path, a Corollary follows.

Corollary 1 *Consider an assembly problem with exactly one critical seller. The buyer cannot realize more than* $\frac{1}{1+\delta}$ *of full surplus in any equilibrium.*

The final result provides a bound on buyer's equilibrium surplus share when the underlying graph contains more than one critical sellers. This result also holds for arbitrary valuations.

Theorem 4 *Consider an assembly problem with 2 or more critical sellers. The buyer's share of surplus cannot exceed* $\frac{1-\delta}{1+\delta}$ *in any equilibrium.*

The benchmark model of Roy Chowdhury and Sengupta (2012) (Section 4.2) has an equilibrium where this bound is achieved exactly. Our proof, given in Appendix D, uses method of contradiction. We show that if such an outcome were possible, one of the critical sellers realizing a surplus share less than $\frac{1}{1+\delta}$ has a profitable deviation. Hence this case cannot be an equilibrium outcome.

The remark on inefficiency of the possible equilibrium outcomes applies here as well. The following Corollary is in the spirit of the previous one.

Corollary 2 *Consider an assembly problem with 2 or more critical sellers. The buyer cannot realize more than* $\frac{1-\delta}{1+\delta}$ *of full surplus in any equilibrium.*

6 Discussion

Our first result claims that if valuations of the sellers are identical and the underlying graph structure does not have a critical seller, there exist equilibria where the buyer extracts full surplus within two periods. Here we have considered the simple advantages of position that certain sellers exact in a graph, and abstracted from advantages due to differences in seller valuations.

We have considered four mutually exclusive and exhaustive categories of graphs, viz., (a) graphs containing cycles of order $k+1$, (b) graphs with two disjoint paths, (c) graphs with critical sellers, and (d) oddball graphs where (i) there is no cycle of length $k+1$, (ii) no two paths are disjoint and (iii) the intersection of all feasible paths is empty. These categories can be easily interpreted in terms of complementarity and substitutability as we have done in Section 4. Of particular interest is the $k+1$ cycle, where every item on a feasible path can be completely substituted by another item on the graph: only in this case, the buyer is able to extract full surplus in the first period, regardless of whether the buyer makes the first offer or the sellers. In other words, in this case, no seller has any positional advantage. Thus, it is comparable to the pure Bertrand competition visible in Example 1. At the other extreme is the graph with critical sellers: such critical sellers exhibit full positional advantage and prevent the buyer from extracting surplus beyond a point, regardless of whoever makes the first offer. Such sellers exhibit full complementarity with respect to any feasible path on the graph.

The cases of graphs with disjoint feasible paths and oddball graphs lie between these two extremes. If the buyer picks a feasible path on any of these graphs, its nodes have limited substitutability. Note that our bargaining protocol only permits cash offers with full commitment. Consequently, once the buyer commits to a seller on a feasible path, she tends to commit to all sellers on that feasible path. Thus, the buyer has to cough up positive shares of the surplus if she is making the first offer. However, the buyer can avoid this commitment problem by making negative offers to all sellers and pushing towards Bertrand competition in the second period. For a patient buyer, the loss of surplus by shifting the onus of bargaining to the sellers is not significant.

Our second result shows that full surplus extraction is not robust with respect to changes in the valuation structure. The buyer cannot extract full surplus when the valuation profile of the sellers is asymmetric, even when no seller is critical. However, the loss in buyer's equilibrium surplus may not be pronounced when the difference between the two lowest sums of valuations across feasible paths ($\mathcal{S}_1$ and $\mathcal{S}_2$) is not large. In this sense, the degree of asymmetry in seller valuations may be of less concern to the buyer than the criticality of certain sellers.

Theorem 4 shows that the bound provided by Roy Chowdhury and Sengupta (2012) on the equilibrium surplus share the buyer can extract in an assembly with two or more critical sellers carries over to our generalized structure. But note that Theorem 3 highlights that there is possibility of higher surplus when only one of the sellers is critical.

To summarize, the possibility of holdout in our model of non-cooperative bargaining depends on the degree of complementarity of items and the degree of asymmetry in the valuations of sellers. We have modelled complementarity through graphs and allowed asymmetry of seller valuations. Our contribution characterizes the extent of holdout in terms of buyer's equilibrium surplus share over the entire spectrum of complementarity and valuation asymmetry.

An obvious extension of this exercise is to investigate the impact of coalition formation among sellers on the surplus shares. For example, consider the problem where one item is required and two sellers are present. First, notice that by making alternate offers to one of the sellers according to the equilibrium strategy specified by Rubinstein (1982) and by excluding the other seller using negative offers, the buyer can assure herself $\frac{\delta}{1+\delta}$ share of full surplus. If we allow sellers to use trigger strategies, there exists an equilibrium where both sellers collude to claim $\frac{1}{1+\delta}$ of the full surplus and the buyer picks one of them with equal probability provided $\delta > \frac{1}{\sqrt{2}}$. This equilibrium is sustained by the following trigger strategy: if any seller deviates by charging less than $\frac{1}{1+\delta}$, the other seller charges zero surplus share in the subsequent period. The buyer then rejects the deviating seller's offer and chooses to purchase from the other seller. The collusive payoff $\frac{1}{2(1+\delta)}$ is greater than the non-collusive payoff $1 - \delta$ if $\delta > \frac{1}{\sqrt{2}}$. In this equilibrium, both sellers gets positive expected payoff. If $\delta < \frac{1}{\sqrt{2}}$, sellers compete and earn zero surplus shares in the equilibrium. A complete investigation of seller coalitions is beyond the scope of current paper.

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A Proof of Theorem 1

Propositions 1-3 constitute the if part of the argument. They cover the three mutually exclusive and exhaustive classes of graphs where no critical seller exists. In all these cases valuations of sellers are identical, and therefore, all feasible paths have the same sum of seller valuations. In each of these cases we provide equilibrium strategies such that the buyer extracts full surplus within two periods. The arguments of these claims are generalizable to graphs with two or more feasible paths corresponding to the minimum sum of valuations. Propositions 4 and 5 constitute the only if part. Proposition 4 covers the classes of graphs where critical sellers exist and Proposition 5 covers the classes of graphs where only one feasible path has the minimum sum of seller valuations. In both these cases we show that there cannot be an equilibrium where the buyer extracts full surplus.

Proposition 1 *Consider an assembly problem* $\langle \Gamma^{\triangle}, k, v, \delta \rangle$ *such that* $v_1 = \cdots = v_n, v_0 > kv_1$*. The buyer extracting full surplus is an equilibrium outcome.*

Proof: We will prove the case of a cycle of length $k+1$. It follows immediately that such an equilibrium can be obtained for any graph containing a cycle of length $k+1$ as a subgraph.

Consider the following strategy of the buyer. She picks a feasible path. Consider any continuation game where $m < k$ plots have already been acquired. If the buyer is making offers, she offers $k-m$ sellers zero surplus and make negative offers to the remaining seller. If sellers are making offers, buyer accepts the lowest $k-m$ claims provided her share of the surplus is non-negative and reject all other claims. If more than $k-m$ sellers are making identical lowest offers, she accepts $k-m$ offers with equal probability.

We claim that given the above strategy, all active sellers claiming zero surplus at any continuation game they are required to make an offer is a best response. Let x_i be the surplus claim of active seller i. No seller can gain by deviating for one stage when $x_i = x_j = 0, i \neq j$. Hence it is an equilibrium. If $x_i = x_j > 0, i \neq j$, either seller i or j can reduce his claim by a small amount and make a gain. If $x_i > x_j \geq 0, i \neq j$, then seller j can increase his claim by a small amount and make a gain. Hence these are not equilibrium claims.

At any continuation where the buyer is making an offer and m plots have already been acquired, the active seller who is made a negative offer rejects it. Simultaneously, $k - m$ sellers immediately accept corresponding zero offers. If any of these sellers reject such offers, they reach a continuation game where the maximum they can gain by rejecting buyer's offers is zero. Hence this is an equilibrium. Trade takes place in the first period itself when $m = 0$, with k sellers who are made zero surplus offers. ■

Since any complete graph of order $n > k$ contains a cycle of length $k + 1$, we get the following Corollary.

COROLLARY **3** *Consider an assembly problem $\langle n, k, v, \delta \rangle$ such that $v_1 = \cdots = v_n, v_0 > kv_1$. The buyer extracting full surplus is an equilibrium outcome.*

REMARK **3** *Note that the equilibrium outcome does not change whether the buyer moves first, or the sellers.*

REMARK **4** Notice that in this case the existence of equilibrium with full surplus extraction is not dependent on the magnitude of the discount factor δ.

REMARK **5** When $k = 2$, then the above result is also true for any graph containing a cycle of length more than 3. But it is not true for $k > 2$. For instance, consider the cycle of length 5 when $k = 3$ (see Figure 6). Suppose the buyer wants to make offers that are acceptable to sellers 1,2 and 3 in the first period itself. Sellers 1 and 3 will accept a zero surplus offer since if they reject, they have to compete with sellers 5 or 4. Seller 2, on the other hand, will not accept a surplus of less than δv_1, since if he rejects an offer, he has to compete with sellers 4 and 5 together. Therefore, the buyer has two ways to complete the transaction in the first period: either (i) she makes zero surplus offers to 4 sellers on the graph and makes a negative offer to the remaining seller, or (ii) she makes zero surplus offers to sellers 1 and 3, make a surplus offer of δv_1 to seller 2, and negative offers to the remaining sellers.

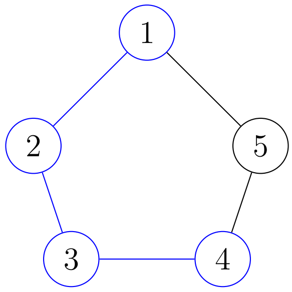

Figure 6: A cycle of length 5; Γ^{SO} in blue

Proposition **2** *Consider an assembly problem* $\langle \Gamma^D, k, v, \delta \rangle$ *such that* $v_1 = \cdots = v_n, v_0 > kv_1$. *(a) If the sellers move first, the buyer extracts full surplus in the first period. (b) If the buyer moves first, there exists* $\overline{\delta}$ *such that* $\forall \delta > \overline{\delta}$ *there is an equilibrium where the buyer extracts full surplus in the second period*

Proof: Consider the following strategy of the buyer: In any continuation game where the buyer makes the offers, she makes negative offers to all sellers. In any continuation game where sellers have the first move, the buyer accepts the claims of sellers on a path with the lowest sum of claims provided her share of surplus is non-negative and reject all other claims. If the sum of claims on the two feasible paths are same, she accepts claims from one of the paths with equal probability.

We claim that given the above strategy, sellers in the two disjoint feasible paths claiming zero surplus at any continuation game they are required to make an offer is a best response. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two feasible paths in Γ^D. Let x_i be the surplus claim of active seller i. No seller can gain by deviating for one stage when $\sum_{i \in \mathcal{P}_1} x_i = \sum_{i \in \mathcal{P}_2} x_i = 0$. Hence it is an equilibrium. If $\sum_{i \in \mathcal{P}_1} x_i = \sum_{i \in \mathcal{P}_2} x_i > 0$, a seller on one of the paths can reduce his claim by a small amount and make a gain. If $\sum_{i \in \mathcal{P}_1} x_i > \sum_{i \in \mathcal{P}_2} x_i$, then any seller on $\mathcal{P}_2$ can increase his claim by a small amount and make a gain. Hence these are not equilibrium claims.

Part (a) of the claim follows immediately. For part (b), note that buyer can make zero surplus offers to sellers on both paths, and negative surplus offers to all other sellers; sellers on both paths would accept these offers. To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - 2kv_1}{v_0 - kv_1}$. The buyer can also make acceptable offers of surplus shares, $\delta(k-1)v_1$, to each seller on one path and negative offers to all other sellers, provided $v_0 - kv_1 - \delta k(k-1)v_1 > 0$. This is because, by rejecting a first period offer from the buyer, a seller on the chosen path competes with sellers on the other path; the highest surplus he can claim in a continuation game where he and the other sellers are making offers is $(k-1)v_1$.To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - kv_1 - \delta k(k-1)v_1}{v_0 - kv_1}$. Thus, provided $\delta > \max\{\frac{v_0-2kv_1}{v_0-kv_1}, \frac{v_0-kv_1-\delta k(k-1)v_1}{v_0-kv_1}\}$, the buyer extracting full surplus in the second period is an equilibrium outcome in the strategies described above. ■

Remark **6** If $\delta \leq \max\{\frac{v_0-2kv_1}{v_0-kv_1}, \frac{v_0-kv_1-\delta k(k-1)v_1}{v_0-kv_1}\}$ then full surplus extraction is not possible in the equilibrium. Either the buyer purchases items from all sellers on a single path by paying positive surplus shares; or, she purchases $2k$ items by offering zero surplus shares to all sellers on two disjoint paths.

Proposition **3** *Consider an assembly problem $\langle \Gamma^O, k, v, \delta \rangle$ such that $v_1 = \cdots = v_n, v_0 > kv_1$. (a) If the sellers move first, the buyer extracts full surplus in the first period. (b) If the buyer moves first, there exists $\overline{\delta}$ such that $\forall \delta > \overline{\delta}$ there is an equilibrium where the buyer extracts full surplus in the second period.*

Proof: We introduce some notation in the next two paragraphs that would be useful in proving the next result.

We note that each graph Γ^O has a subgraph Γ^{SO} such that (i) it contains a feasible path $\mathcal{P}$, (ii) for each node $x \in \mathcal{P}$ there exists a node $y \in \Gamma^O - \Gamma^{SO}$ and an edge $e(y,z), z \in \Gamma^{SO}$ such that $\Gamma^{SO} - x + z$ contains a feasible path of length k. For instance, in Figure 6, the path $\{1234\}$ qualifies as Γ^{SO}. Figure 5 shows one more example. Observe that the order of any Γ^{SO} would vary from k to $n-1$. For any given Γ^O, let Γ^{SO*} be the smallest of all $\Gamma^{SO} \subset \Gamma^O$ with order m^*.

Further, pick any feasible path $\mathcal{P}$ of length k on Γ^O. For each x on $\mathcal{P}$, let $s(x)$ be the order of the smallest subgraph Γ_S of Γ^O such that $(\mathcal{P} - x) \cup \Gamma^S$ is a feasible path of length k. For example, in Figure 5, $s(1) = s(5) = 1$ and $s(2) = 2$.

Consider the following strategy of the buyer: In any continuation game where the buyer has the first move , the buyer makes negative offers to all sellers. In any continuation game where sellers have the first move, the buyer accepts the claims of sellers on a path with the lowest sum of claims provided her share of surplus is non-negative and reject all other claims. In case the sum of claims on the two feasible paths are same, she accepts claims from one of the paths chosen with equal probability.

We claim that given the above strategy, sellers claiming zero surplus at any subgame they are required to make an offer is a best response. Let $\mathcal{P}_1, \ldots \mathcal{P}_m$ be the feasible paths in Γ^O. Let x_i be the surplus claim of active seller i. No seller can gain by deviating for one stage when $\sum_{i \in \mathcal{P}_1} x_i = \cdots = \sum_{i \in \mathcal{P}_m} x_i = 0$. This is because, for each x_i, there is always a feasible path in Γ^O that does not contain x_i. Hence it is an equilibrium. If $\sum_{i\in\mathcal{P}_1} x_i = \cdots = \sum_{i\in\mathcal{P}_m} x_i > 0$, a seller on either path can reduce his claim by a small amount and make a gain. If $\sum_{i\in\mathcal{P}_1} x_i > \sum_{i\in\mathcal{P}_2} x_i$, then any seller on $\mathcal{P}_2$ can increase his claim by a small amount and make a gain. Hence these are not equilibrium claims.

Part (a) of the claim follows immediately. For part (b), note that buyer can make zero surplus offers to all sellers on Γ^{SO*}, and negative surplus offers to all other sellers; sellers on Γ^{SO*} would accept these offers. To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - m^* v_1}{v_0 - kv_1}$. The buyer can also make acceptable offers of surplus shares to sellers on a path and negative offers to all other sellers. If $\mathcal{P}$ is the picked path and x_i is the node corresponding to seller i, he accepts any surplus share at least equal to $\delta(s(x_i) - 1)v_1$. This is possible when $v_0 - kv_1 - \delta \sum_{i\in\mathcal{P}}(s(x_i) - 1)v_1 > 0$. This is because, by rejecting a first period offer from the buyer, a seller on the chosen path competes with sellers on the other path; the highest surplus he can claim in a continuation game where he and the other sellers are making offers is $(s(x_i) - 1)v_1$.To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - kv_1 - \delta \sum_{i\in\mathcal{P}}(s(x_i)-1)v_1}{v_0 - kv_1}$. Thus, provided $\delta > \max\{\frac{v_0 - m^* v_1}{v_0 - kv_1}, \frac{v_0 - kv_1 - \delta \sum_{i\in\mathcal{P}}(s(x_i)-1)v_1}{v_0 - kv_1}\}$, the buyer extracting full surplus in the second period is an equilibrium outcome in the strategies described above. ■

Remark 7 If $\delta < \frac{v_0 - kv_1 - \delta \sum_{i\in\mathcal{P}}(s(x_i)-1)v_1}{v_0 - kv_1}$. Thus, provided $\delta > \max\{\frac{v_0 - m^* v_1}{v_0 - kv_1}, \frac{v_0 - kv_1 - \delta \sum_{i\in\mathcal{P}}(s(x_i)-1)v_1}{v_0 - kv_1}\}$ then full surplus extraction is not possible in the equilibrium: either buyer purchases items from all sellers on a single path by paying positive surplus shares; or, she purchases more than k items by offering zero surplus shares to corresponding sellers.

Remark 8 Observe that in Propositions 1-3 the underlying graph does not contain a critical seller. If the graph has only one component, then the corresponding bargaining game has an equilibrium where the buyer extracts full surplus in at most two periods. Suppose the graph has multiple components. When seller valuations are identical, there exists an equilibrium where the buyer extracts full surplus in the first period itself if and only if the graph contains a $k+1$-cycle. Otherwise, (a)competing paths lie in different components or (b) form an oddball graph. In these cases, there exist an equilibrium where the buyer extracts full surplus in the second period if she is making the first set of offers, or in the first period itself if sellers are making the first set of offers.

Now consider assembly problems where the underlying graph contains at least one critical seller. The following result is obtained without any assumption on valuations.

Proposition 4 *Suppose $\Gamma = \Gamma^*$. The buyer cannot extract full surplus in an equilibrium.*

Proof: Suppose there is an equilibrium where the buyer obtains full surplus. We will show that in any such equilibrium a critical seller has profitable deviation.

If $k = 1$, then there is a unique equilibrium by Rubinstein (1982) where buyer gets $\frac{1}{1+\delta}$ if offering first and $\frac{\delta}{1+\delta}$ if the seller is offering first. Suppose $k \geq 2$. Suppose, if possible, that the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on an efficient path are selling their items at period t or some period before t. Consider a critical seller whom trades at period $\hat{t}$ at zero surplus share. Now by rejecting buyers zero offer or by quoting an offer unacceptable to the buyer, a critical seller moves to a continuation game in period $t+1$ where he is the only active seller. By the result due to Rubinstein (1982) (see Section 4.2) the critical seller obtains a positive surplus share in the continuation game.This constitutes a profitable deviation for the critical seller. ■

Remark 9 The proof above implies that in any equilibrium of $\Gamma = \Gamma^*$ at least one critical seller earns a positive surplus share. The non-critical sellers may or may not get positive shares.

Now consider the case where seller valuations are not equal. In this case, the sum of seller valuations may differ over paths. The path corresponding to the least sum of seller valuations is efficient in the sense that it corresponds to highest potential surplus. It follows that if possible, the buyer would prefer to purchase the efficient path.

Let $\mathcal{P}_i$ denote the path corresponding to the i-th smallest sum of valuations on a path in Γ. We will refer to a set of assembly problems as *rich* if there does not exist two distinct paths $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{S}_1 = \mathcal{S}_2$. Suppose the richness condition is not satisfied. Consider the following strategy for the buyer: she offers negative surplus shares to all sellers whenever she is supposed to make an offer, and agrees to trade with sellers on the path with lowest sum of valuations provided it leads to a positive surplus. If the buyer is proposing first, all sellers reject buyer's offers. In the next period, sellers on $\mathcal{P}_1$ and $\mathcal{P}_2$ cannot claim any surplus: the buyer extracts full surplus in the second period. If the sellers are making offers first, sellers on these two paths cannot claim any surplus share.

Proposition 5 *Consider the rich class of assembly problems $\langle \Gamma, k, v, \delta \rangle$ such that $v_1 \leq \cdots \leq v_n$ with at least one strict inequality. There does not exist any equilibrium where the buyer extracts full surplus.*

Proof: **Case 1** ($\Gamma = \Gamma^{\Delta}$): Suppose the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on the efficient path sell their items at t or prior to t. Let us pick a seller i on the efficient path and suppose he is last active at period $\hat{t} \leqslant t$. Since buyer extracts full surplus in the proposed equilibrium either i proposes zero surplus share at $\hat{t}$ or accepts a zero surplus share offer at $\hat{t}$. Now pick the seller j who is on $\mathcal{P}_2$ but not on $\mathcal{P}_1$. Since by assumption $\mathcal{S}_1 < \mathcal{S}_2$, $v_i < v_j$. Consider the following deviation for seller i at $\hat{t}$: i makes a surplus offer of $v_j - v_i - \epsilon$ and accepts offer greater than $\delta(v_j - v_i - \epsilon)$. Here ϵ is a small positive quantity. If the buyer keeps rejecting i offer and keeps offering less than the claim of i then we reach a continuation game where i is the only active seller on the efficient path. Note that for small ϵ, buyer would never agree to trade with seller j. In this continuation game i can ensure a positive surplus.
Case 2 ($\Gamma = \Gamma^{D}$): Suppose the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on the efficient path sell their items at t or prior to t. Let us pick a seller i on the efficient path and suppose he is last active at period $\hat{t} \leqslant t$. Since buyer extracts full surplus in the proposed equilibrium either i proposes zero surplus share at $\hat{t}$ or accepts a zero surplus share offer at $\hat{t}$. Now consider the deviation strategy of i where he makes an offer of $\mathcal{S}_2 - v_i - \epsilon$ and accept offers of at least $\delta(\mathcal{S}_2 - v_i - \epsilon)$. Then there exists a continuation game at period $t+1$ where i is the only remaining active seller on the efficient path and can guarantee himself a positive payoff.
Case 3 ($\Gamma = \Gamma^{O}$): Suppose the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on the efficient path sell their items at t or prior to t.Let us pick a seller i at the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$ and suppose he is last active at period $\hat{t} \leqslant t$.Since buyer extracts full surplus in the proposed equilibrium either i proposes zero surplus share at $\hat{t}$ or accepts a zero surplus share offer at $\hat{t}$. Now consider a deviation strategy for seller i: suppose the cheapest path on the subgraph excluding i is $\mathcal{P}_R$ and the corresponding sum of valuations is $\mathcal{S}_R$; seller i claims $\mathcal{S}_R - \mathcal{S}_1 - \epsilon$ and accepts no less than $\delta(\mathcal{S}_R - \mathcal{S}_1 - \epsilon)$. In this case there is a continuation game at $t+1$ where i is the only active seller on the efficient path and can guarantee himself a positive surplus. ■

B Proof of Theorem 2

We prove by contradiction. Suppose, if possible, the buyer realizes the highest possible surplus $X > v_0 - \mathcal{S}_2$ in an equilibrium, completing trading at period t. It follows that the buyer purchases the efficient path in this equilibrium, since purchasing any other path would imply that the total surplus realized can be at most $v_0 - \mathcal{S}_2$. This implies that there exists at least one seller i in $\mathcal{P}_1$ who is claiming a surplus share x such that $v_i + x$ is less than the valuation v_j of some seller j in $\mathcal{P}_2$.

Suppose, if possible, this seller i trades at t. Consider the following deviation strategy for player i at t, which we call an ϵ-deviation. He can claim an increment $\epsilon < v_j - v_i$ on his surplus share in the continuation game beginning period t. By accepting this offer, the buyer would realize $X - \epsilon$, while by rejecting, she would get at most δX. For $\epsilon < (1-\delta)X$, buyer would accept the claim of i. If the buyer is making an offer of surplus share x such that $v_i + x < b_j$ at t, seller i can reject this offer and claim a surplus share $\frac{x}{\delta} + \epsilon$ at $t + 1$. If the buyer rejects this offer, her share can be at most δX in the continuation game, whereas by accepting, she gets $X + x - \frac{x}{\delta} - \epsilon$. For $\epsilon < (1 - \delta)(X - \frac{x}{\delta}$, the buyer must accept this offer. Notice that the buyer cannot preserve her surplus in response to such deviations by changing the offers made to other sellers: if it were possible in the continuation game, it is also possible to increase buyer's surplus at t when the seller does not deviate, hence it cannot be an equilibrium. Thus no seller trading at period t can be claiming a price less than the valuation of some seller j in $\mathcal{P}_2$.

Now suppose, if possible, that this seller i is making offers at $t - 1$. Suppose the seller applies the ϵ deviation. Then either the buyer agrees to trade at the new claim, or the seller moves to period t, where he cannot be getting a price less than the valuation of some seller j in $\mathcal{P}_2$. The buyer would agree to trade at the new claim as long as $X - \epsilon > \max(\delta X, X + x - (b_j - v_i)$. If the buyer is making an offer of surplus share x such that $v_i + x < b_j$ at $t - 1$, seller i can reject this offer and claim $\frac{x}{\delta} + \epsilon$ at t. If the buyer rejects this offer, her share can be at most δX in the continuation game, whereas by accepting, she gets $X + x - \frac{x}{\delta} - \epsilon$. For $\epsilon < (1 - \delta)(X - \frac{x}{\delta}$, the buyer must accept this offer.

The argument is then completed by applying an induction on t. Suppose the ϵ-deviation is profitable for seller i at periods $t, t - 1, \ldots, t - l$. Applying the argument of previous paragraph again, we can show that the ϵ-deviation is profitable for seller i at periods $t - l - 1$. This completes the proof.

C Proof of Theorem 3

We prove by contradiction. If possible, suppose there exists an equilibrium where the buyer gets a surplus share strictly higher than $\frac{1}{1+\delta}$. Since the buyer realizes a strictly positive surplus, the game terminates at some finite period t. This implies that the critical seller gets a surplus share strictly less than $\frac{\delta}{1+\delta}$. We show a profitable deviation strategy for the critical seller. Note that no other seller changes his strategy. Consequently, the buyer continues to trade with the same sellers in the continuation game at the same prices as agreed upon in the equilibrium proposed. It is not possible for the buyer to trade with other sellers or with the same sellers at lower prices and increase her payoff in the continuation game: if it were possible in the continuation game, then it is also possible in the original configuration, which therefore is no longer an equilibrium.

Suppose the critical seller agrees to trade at t. If $k = 1$, by Rubinstein (1982), in the continuation game beginning at t, he gets a surplus share equal to $\frac{\delta}{1+\delta}$ if the buyer is making an offer and $\frac{1}{1+\delta}$ if himself making an offer. Consequently, buyer's surplus share cannot exceed $\frac{1}{1+\delta}$. Suppose $k \geq 2$ and if possible, buyer's share is the highest $X > \frac{1}{1+\delta}$. Then the critical seller is getting a share $x < \frac{\delta}{1+\delta}$. We show that this cannot be an equilibrium. Suppose the critical seller trades at t: if buyer is making an offer $x < \frac{\delta}{1+\delta}$ at t, the seller can reject and claim $\frac{x}{\delta} + \epsilon$ at $t + 1$. If the buyer rejects this offer, her share can be at most δX in the continuation game, whereas she gets $X + x - \frac{x}{\delta} - \epsilon$ at $t + 1$ by accepting this offer. For $\epsilon < (1 - \delta) \left(X - \frac{x}{\delta}\right)$, the buyer must accept this offer. If the seller is making an offer $x < \frac{\delta}{1+\delta}$, she can claim an increment of ϵ on this offer: if the buyer accepts, she realizes $X - \epsilon$ at t, whereas by rejecting she can get at most δX in the continuation game beginning at $t + 1$. For $\epsilon < (1 - \delta)X$, she will accept this offer.

Suppose the critical seller trades at $t-1$, : if buyer is making an offer $x < \frac{\delta}{1+\delta}$ at $t-1$, the seller can reject and claim $\frac{x}{\delta} + \epsilon$ at t. If the buyer rejects this offer, her maximum possible share is δX in the continuation game, whereas she gets $X + x - \frac{x}{\delta} - \epsilon$ at t by accepting this offer. For $\epsilon < (1 - \delta) \left(X - \frac{x}{\delta}\right)$, the buyer must accept this offer. If the seller is making an offer $x < \frac{\delta}{1+\delta}$, she can claim an increment of ϵ on this offer: if the buyer accepts, she realizes $X - \epsilon$, whereas by rejecting she can get at most $\max\left(\frac{1}{1+\delta}, \delta X\right)$ in the continuation game beginning at t. For $\epsilon < X - \max\left(\frac{1}{1+\delta}, \delta X\right)$, she will accept this offer.

Suppose the following claim is true: if the critical seller trades at $t-s$, $s \in \{1, 2 \ldots, \}$, buyer's surplus share cannot be more than $\frac{1}{1+\delta}$. We argue that it is true for $t-(s+1)$ also : if buyer is making an offer $x < \frac{\delta}{1+\delta}$ at $t-s-1$, the seller can reject and claim $\frac{x}{\delta}+\epsilon$ at $t-s$. If the buyer rejects this offer, her share is $\max\left(\frac{1}{1+\delta}, \delta X\right)$ in the continuation game, whereas she gets $X+x-\frac{x}{\delta}-\epsilon$ at $t-s$ by accepting this offer. For $\epsilon < X-\max\left(\frac{1}{1+\delta}, \delta X\right)-(1-\delta)\frac{x}{\delta}$, the buyer must accept this offer. If the seller is making an offer $x < \frac{\delta}{1+\delta}$, she can claim an increment of ϵ on this offer: if the buyer accepts, she realizes $X - \epsilon$ at $t - s - 1$, whereas by rejecting she can get at most $\max\left(\frac{1}{1+\delta}, \delta X\right)$ in the continuation game beginning at $t - s$. For $\epsilon < X - \max\left(\frac{1}{1+\delta}, \delta X\right)$, she will accept this offer. Since this is true for $s = 1$, this completes the proof by induction.

D Proof of Theorem 4

We show a profitable deviation strategy for at least one critical seller. As noted in the proof of Theorem 3 above, the buyer continues to trade with the same sellers in the continuation game at the same prices as agreed upon in the equilibrium proposed.

Consider an equilibrium in an assembly problem with $k \geq m \geq 2$ critical sellers. If $k = m$, then by Roy Chowdhury and Sengupta (2012) (see Section 4.2), buyer's surplus share cannot exceed $\frac{1-\delta}{1+\delta}$. Suppose $k > m$. Suppose the final agreement takes place at period t.

Step 1: Suppose trade takes place with at least one critical seller at t and buyer is getting her highest surplus share $X > \frac{1-\delta}{1+\delta}$. We argue that critical sellers trading at t cannot be getting $x < \frac{\delta}{1+\delta}$. If buyer is making an offer at t, a critical seller can reject buyer's offer and successfully claim $\frac{x}{\delta}+\epsilon$ in the next period for a small $\epsilon < (1-\delta)\left(X - \frac{x}{\delta}\right)$. If the sellers are making an offer, then if this critical seller claims an increment ϵ over x, then the buyer accepts as long as $X - \epsilon > \delta X$. It also implies that if two or more critical sellers trade at t, buyer's surplus cannot be more than $\frac{1-\delta}{1+\delta}$.

Step 2: Now suppose no critical seller trades at t while at least one critical seller trades at $t-1$ and buyer is getting her highest surplus share $X > \frac{1-\delta}{1+\delta}$. We argue that critical sellers trading at $t-1$ cannot be getting $x < \frac{\delta}{1+\delta}$. If buyer is making an offer at $t-1$, this critical seller can reject buyer's offer and claim $\frac{x}{\delta} + \epsilon$ at t. Buyer accepts this claim for a small $\epsilon < (1-\delta)\left(X - \frac{x}{\delta}\right)$. If the sellers are making an offer, then if this critical seller claims an increment ϵ over x, then the buyer accepts as long as $x + \epsilon < \frac{\delta}{1+\delta}$. It also implies that if two or more critical sellers trade at $t-1$ but none at t, buyer's surplus cannot be more than $\frac{1-\delta}{1+\delta}$.

Step 3: Now suppose no critical seller trades at t, at least one critical seller trades at $t-1$, some critical sellers trade at some period $t-s-1$, $s \in \{1, 2, \ldots\}$, and buyer is getting her highest surplus share $X > \frac{1-\delta}{1+\delta}$. We know by Step 2 that critical sellers trading at $t-1$ cannot be getting $x < \frac{\delta}{1+\delta}$. So if two or more critical sellers trade at $t-1$, buyer cannot realize more than $\frac{1-\delta}{1+\delta}$. So suppose exactly one critical seller trades at $t-1$, and rest trade at $t-2$. If buyer is getting X, then at least one critical seller is trading at $x < \frac{\delta}{1+\delta}$ at $t-2$. If buyer is making an offer at $t-2$, this critical seller can reject buyer's offer and claim $\frac{x}{\delta} + \epsilon$ at $t-1$. Buyer accepts this claim for a small $\epsilon < X - \max\left(\frac{1-\delta}{1+\delta}, \delta X\right) + \frac{(1-\delta)x}{\delta}$. If the sellers are making an offer, then if this critical seller claims an increment ϵ over x, then the buyer accepts as long as $X - \epsilon < \max\left(\frac{1-\delta}{1+\delta}, \delta X\right)$. Hence this cannot be an equilibrium. Thus the claim is true for $s = 1$. Suppose the claim is true for $s = 1, \ldots, l$, then it must be true for $s = l + 1$: if buyer is realizing X, then at least one seller getting $x < \frac{\delta}{1+\delta}$ has a similar profitable deviation as shown above.

Step 4: Now suppose some critical sellers trade at t while others trade at $t-1$ and buyer is getting her highest surplus share $X > \frac{1-\delta}{1+\delta}$. We know by Step 1 that critical sellers trading at t must be getting at least $\frac{\delta}{1+\delta}$. We now argue that at least one critical seller trading at $t-1$ cannot be getting $x < \frac{\delta}{1+\delta}$. If buyer is making an offer at $t-1$, such a critical seller can reject buyer's offer and claim $\frac{x}{\delta} + \epsilon$ at t. Buyer accepts this claim for a small $\epsilon < (1-\delta)\left(X - \frac{x}{\delta}\right)$. If the sellers are making an offer, then if this critical seller claims an increment ϵ over x, then the buyer accepts as long as $X - \epsilon < \max\left(\frac{1-\delta}{1+\delta}, \delta X\right)$.

Step 5: Now suppose some critical sellers trade at t while others trade at $t-s$, $s \geq 1$. We claim that buyer cannot be getting a surplus share greater than $\frac{1-\delta}{1+\delta}$. We know by Step 3 that the claim is true for $s = 1$. We now use an induction argument. Suppose the claim is true for $s = 1, 2, \ldots, l$. We show that the claim must be true for $s = l+1$. By Step 1, no critical seller trading at t must be getting less than $\frac{\delta}{1+\delta}$. So, if there are at least two critical sellers trading at t, then the claim is true. So suppose only one critical seller is trading at t and getting at least $\frac{\delta}{1+\delta}$ while others trade at $t-l-1$. If the claim is false, then a critical seller is trading at $x < \frac{\delta}{1+\delta}$ at $t-l-1$. If buyer is making an offer at $t-l$, such a critical seller can reject buyer's offer and claim $\frac{x}{\delta} + \epsilon$ at $t-l$. Buyer accepts this claim for a small $\epsilon < X - \left(\frac{1-\delta}{1+\delta}, \delta X\right) + \frac{(1-\delta)x}{\delta}$. If the sellers are making an offer, then if this critical seller claims an increment ϵ over x, then the buyer accepts as long as $X - \epsilon < \max\left(\frac{1-\delta}{1+\delta}, \delta X\right)$. Hence this cannot be an equilibrium.

Step 6: Suppose critical sellers trade for the last time at $t-s$, $s \in \{1, 2, \ldots\}$, while others trade at $s' < t-s$, and so on. We claim that buyer's equlibrium surplus share cannot be more than $\frac{1-\delta}{1+\delta}$. We know that the claim is true for $s = 1$ by Step 3. Suppose the claim is true for $s = 1, \ldots l$. We claim that it must be true for $s = l+1$. Suppose buyer is realizing her highest surplus share $X > \frac{1-\delta}{1+\delta}$. First notice that critical sellers trading at $t-l-1$ cannot be getting $x < \frac{\delta}{1+\delta}$. If buyer is making an offer at $t-l-1$, such a critical seller can reject buyer's offer and claim $\frac{x}{\delta} + \epsilon$ at $t-l$. Buyer accepts this claim for a small $\epsilon < X - \left(\frac{1-\delta}{1+\delta}, \delta X\right) + \frac{(1-\delta)x}{\delta}$. If the sellers are making an offer, then if this critical seller claims an increment ϵ over x, then the buyer accepts as long as $X - \epsilon < \max\left(\frac{1-\delta}{1+\delta}, \delta X\right)$. Consequently if there are two or more critical sellers trading at $t-l-1$, then buyer's surplus share cannot be more than $\frac{1-\delta}{1+\delta}$. So suppose that one critical seller trades at $t-l-1$ and at least one critical seller trades at some $t-l-l'$, $l' \in \{2, 3, \ldots\}$. If $l' = 2$, then the critical seller trading at $x < \frac{\delta}{1+\delta}$ has a profitable deviation similar to the one discussed above. An induction on l' then completes the argument.

Steps 1, 4 and 5 prove the claim of the Theorem when trade last occurs with critical sellers at t, while Steps 2, 3 and 6 prove the claim for the case when trade last occurs with critical sellers at some period prior to t.