Classical Optical Modelling of the 'Prisoner's Dilemma' Game

Preprint · May 2020

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Classical Optical Modelling of the ‘Prisoner’s Dilemma’ Game

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Abstract

Though there is a strong body of literature on ‘pure quantum games’ where pure quantum entanglement needs to be introduced in the modelling for Pareto Superior equilibria, unattainable in classical games, we demonstrate in the current paper that similar, though slightly different, results can also be obtained through classical optical modelling of normal games. The phenomenon of entanglement in classical polarization optics can be utilized in designing non-zero sum cooperation games to attain superior equilibria, the advantage of such designs being ease of implementation and robustness of entanglement, unlike fragile quantum entanglement due to decoherence. We demonstrate the foundational set up of classical optical modelling of game theory by using the basic Prisoners’ Dilemma game under full information.

Keywords: : classical optical modelling (COM), entanglement, Prisoners Dilemma (PD), Pareto Superiority, joint probabilities, non-zero-sum games

1 Introduction

It was John von Neuman and Oskar Morgenstern [1] who first developed the mathematical theory of games in the form known to economists, social scientists and biologists. Over the past seven decades the framework has been deepened and generalized. The paradigmatic game that has found widespread applications in economy, psychology, ecology and biology is the Prisoners’ Dilemma which falls in the category of ‘non-zero sum games’ where– in contrast to ‘zero-sum games’– the two players are not in strict opposition to each other, but may rather benefit from mutual cooperation. Recently, some of these games have been ‘quantized’ [2, 3]. Quantum games differ from their classical counterparts in three main

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ways: they admit (i) superposition of initial states, (ii) entanglement of initial states and (iii) superposition of strategies to be used on the initial states.

It is interesting to note that classical polarization optics, as opposed to classical particle mechanics, offers similar advantages to games because of the occurrence of superpositions and entanglement [4, 5, 6, 7, 8, 9]. In fact, classical entanglement has the advantage of being robust, unlike quantum entanglement which is fragile in the presence of noise.

Recently a *Classical Optical Model* (COM) of social sciences has been developed [10] which has striking similarities with 'quantum models' [11, 12, 13]. It has the advantage of being simple, geometric and intuitive, and is based on the epistemological approach of Bohr who adopted a definite Kantian viewpoint with subtle differences.

Prisoners' Dilemma (PD) has become paradigmatic in another sense. In this non-zero sum game, if we start from the assumption that individual players are typically rational and utility maximizing, where the utility function is typically individual, and rationality is 'common knowledge', then instead of a Pareto Superior cooperation strategy, the players are forced to choose an inferior defection strategy. This is a tragic consequence of rationality itself. Though later repeated game or evolutionary Prisoners' Dilemma has been proposed [14] where cooperation might emerge based on 'discount rates' of individual players (we elaborate on this point later in the paper).

Now the advantage of a quantum-like modification of the game is that, there can be additional quantum strategies. Some authors propose a 'super operator' or a 'super strategy' which would converge the game to a Pareto Superior cooperative equilibrium, thus demonstrating 'quantum supremacy' [15]. Hence, if such superior strategies are played or are available to the players, then a good cooperative equilibrium can be established in the first round itself, rather than extending. We think there is another advantage of such an alternative modelling, namely, it is not necessary to modify the underlying utility function structure (as in the behavioural economics and finance literature) by, for example, introducing 'altruism' or 'inequity-aversion' [16]. Rather, here the players are still 'rational' in the neoclassical sense of the word, but the major modifications are in the strategic profiles, 'entangled' or 'correlated' game structure, and decision framework (probability computation using a modified formula for total probability (FTP) and updating of beliefs when required of the players).

Hence overall, our framework would provide: (i) a simple geometric representation of the PD game with Pareto Superior outcomes for 'classical optical' entanglement-like scenario which can be in reality sustained (unlike the very fragile pure quantum entanglement where the no-signaling theorem further prohibits utilizing such entanglement for communication purposes), and (ii) a formula for total probability (FTP) which can be used to demonstrate the deviant behaviours of 'rational' players from the Nash Equilibrium (NE) under scenarios of uncertainty.

The use of the Formula for Total Probability (FTP) needs to be emphasized here. It has been observed that under real life scenarios, for example, when players are in an ambiguity or uncertainty scenario, the moves exhibited by them do not follow dominant Nash Equilibrium, though exact reasons for this deviant behavior may well be behavioral and complex [12]. FTP computed on the basis of Born's rule can provide a description of such deviations due to the presence of additional perturbative terms in the probability computations also known as 'interference' terms. Again, such interference terms would be

absent in classical measure theory-based computations. In the recent development of COM of decision making or cognition [10] a modified formula for total probability has been developed. However, such a formula can be utilized in case of 'sequential' move games, rather than in 'simultaneous' move games, which is the Prisoners' Dilemma in normal form, as will be discussed. In PD games therefore we will use joint probabilities rather than conditional probabilities. In future versions we would expand on 'sequential' move cooperative non-zero sum games.
In the present paper we wish to develop a COM of the Prisoners' Dilemma Game.

2 COM of Prisoners' Dilemma Game

Our starting point will be two players/prisoners Alice (A) and Bob (B) who independently decide whether they choose to defect (strategy D) or cooperate (strategy C). Depending on the decision they take, each player receives a certain pay-off (Table 1). The objective of each player is to maximize his or her individual pay-off. The catch of the dilemma is that D happens to be the 'dominant strategy', i.e. the optimal move for an individual player regardless of how the other players act. Each player rationally and independently chooses to defect, although by doing so, they do worse than if they had both decided to cooperate (see pay-off Table). This is because D is the *Nash Equilibrium* [17] in which none of the players sees any advantage in choosing to act otherwise.
Following closely Ref.[3], we propose a physical model which consists of (a) sources of two polarized classical light beams, one for each player, (b) a set of optical instruments which enable each player to manipulate his or her own light beam in a strategic manner, and (c) a measurement device (polarization analyzers) which determines the players' pay-off according to the polarization states of the light beams detected. All of these are assumed to be perfectly known to both Alice and Bob.
To the possible outcomes of the classical strategies C and D are assigned two vectors $|C\rangle$ and $|D\rangle$ which span the basis of the 2-dimensional Hilbert space $\mathcal{H}$ of a polarized light beam. The state of the game at each instant is described by a vector in the tensor product space $\mathcal{H_A} \otimes \mathcal{H_B}$ of the two light beams spanned by the basis $|C\rangle_A \otimes |C\rangle_B$, $|C\rangle_A \otimes |D\rangle_B$, $|D\rangle_A \otimes |C\rangle_B$ and $|D\rangle_A \otimes |D\rangle_B$.
Let us identify $|C\rangle := |0\rangle$ and $|D\rangle := |1\rangle$ as the standard basis vectors of a Poincaré sphere. They correspond to horizontal (H) and vertical (V) polarization states of light. Let the optical states with Alice and Bob in this basis be

$$|\psi\rangle_i = \cos\theta_i |C\rangle_i + e^{i\phi_i} \sin\theta_i |D\rangle_i, \tag{1}$$

where $i = A, B$ and $-\pi/2 \leq 2\theta_i \leq +\pi/2$ and $0 \leq 2\phi_i \leq 2\pi$. These states are the points $(2\theta, 2\phi)$ on the Poincaré sphere. The 'strategic space' is the family of unitary matrices $U(2)$ [19]

$$\hat{U}(\theta, \phi) = e^{i\phi} \begin{pmatrix} r & it^* \\ it & r^* \end{pmatrix} \tag{2}$$

where ϕ is any phase and (r, t) are parameters satisfying the condition $|r|^2 + |t|^2 = 1$. Using the parametrization $r = \cos\theta e^{-i\phi}, t = \sin\theta e^{-i\phi}$, this can be written in the form

$$\hat{U}(\theta,\phi) = \begin{pmatrix} \cos\theta & i\sin\theta \\ ie^{2i\phi}\sin\theta & e^{2i\phi}\cos\theta \end{pmatrix}. \tag{3}$$

One readily sees that

$$\hat{U}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0 = \mathbb{I}, \tag{4}$$

$$-i\hat{U}\left(\frac{\pi}{2},\pi\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \tag{5}$$

$$-\hat{U}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \tag{6}$$

$$\hat{U}\left(0,\frac{\pi}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \tag{7}$$

where $\sigma_i (i = 1,2,3)$ are the Pauli matrices which are generators of the group $SU(2)$. They satisfy the algebra

$$[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{8}$$

Since these matrices form a closed set of non-trivial operators, one can associate at most three independent strategies with them which may be realized using beam splitters [19]. We shall make the following identifications: 'cooperation $\hat{C} = \sigma_3$, 'defect' $\hat{D} = i\sigma_2$ and 'abstain' $\hat{L} = \sigma_1$.
Let us note in passing here that

$$\frac{1}{\sqrt{2}}[\hat{C}+\hat{L}] = \frac{1}{\sqrt{2}}[\sigma_3+\sigma_1] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} := \hat{H} \tag{9}$$

which is the Hadamard gate, and that the unitary operator $\hat{U}(\theta,\phi)$ can also be written in the form

$$\hat{U}(\theta,\phi) = e^{i\phi}\exp(-i\theta\vec{\sigma}.\vec{n}) \tag{10}$$

which is a rotation by angle 2θ about an arbitrary direction $\vec{n}$ of the Poincaré sphere.
It must be emphasized here that the use of unitary transformations and the Pauli matrices is quite legitimate within a purely classical Hilbert space theory. The algebra (8) of the Pauli matrices is a property of the non-Abelian Lie group $SU(2)$, and Planck's constant does not enter into it.
Denoting Alice's and Bob's strategies by $\hat{U}_A$ and $\hat{U}_B$ respectively, their joint strategy will be $\hat{U}_A \otimes \hat{U}_B$, each one of them acting only on his/her light beam. If the initial state in the tensor product space is $|\psi_0\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, then the final state is $|\psi_f\rangle = \hat{U}_A \otimes \hat{U}_B|\psi_0\rangle$. The subsequent detection of this state by two polarization analyzers (e.g. polarizing beam splitters) will yield particular outcomes, CC, CD etc, depending on the strategies adopted by Alice and Bob. In the standard version of the Prisoners' Paradox game the pay-off is returned according to the corresponding entry of the pay-off matrix (Table 1). Let us consider Alice's and Bob's pay-offs in general,

$$\$_A = rP_{CC} + pP_{DD} + tP_{DC} + sP_{CD}, \tag{11}$$

$$\$_B = rP_{CC} + pP_{DD} + sP_{DC} + tP_{CD} \tag{12}$$

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Table 1: Pay-off Table

	Bob: C	**Bob: D**
Alice: C	(3,3)	(0,5)
Alice: D	(5,0)	(1.1)

Table 1. The first entry in the parentheses is Alice's pay-off and the second entry that of Bob in the standard Prisoners' Paradox game.

where $P_{\lambda\lambda'} = |\langle\lambda\lambda'|\psi_f\rangle|^2$ is the joint probability that the output ports record λ and λ' polarization states. The symbols (r, p, s, t) stand for 'reward', 'punishment', 'temptation' and 'sucker's pay-off', the numerical values being those given in Table 1 (r= 3, p = 1, s = 0, t = 5). Now,

$$\begin{aligned}
|\psi_f\rangle &= \hat{U}_A \otimes \hat{U}_B|\psi_0\rangle \\
&= \cos\theta_A \cos\theta_B \hat{U}_A|C\rangle_A \otimes \hat{U}_B|C\rangle_B + e^{i\phi_B}\cos\theta_A \sin\theta_B \hat{U}_A|C\rangle_A \otimes \hat{U}_B|D\rangle_B \\
&+ e^{i\phi_A}\sin\theta_A \cos\theta_B \hat{U}_A|D\rangle_A \otimes \hat{U}_B|C\rangle_B \\
&+ e^{i(\phi_A+\phi_B)}\sin\theta_A \sin\theta_B \hat{U}_A|D\rangle_A \otimes \hat{U}_B|D\rangle_B. \qquad (13)
\end{aligned}$$

Hence, if both Alice and Bob choose strategy $\hat{C}$, a simple calculation gives

$$\begin{aligned}
P_{CC} &= \cos^2\theta_A \cos^2\theta_B,\ P_{DD} = \sin^2\theta_A \sin^2\theta_B, \\
P_{CD} &= \cos^2\theta_A \sin^2\theta_B,\ P_{DC} = \sin^2\theta_A \cos^2\theta_B. \qquad (14)
\end{aligned}$$

On the other hand, if they both choose strategy $\hat{D}$,

$$\begin{aligned}
P_{CC} &= \sin^2\theta_A \sin^2\theta_B,\ P_{DD} = \cos^2\theta_A \cos^2\theta_B, \\
P_{CD} &= \sin^2\theta_A \cos^2\theta_B,\ P_{DC} = \cos^2\theta_A \sin^2\theta_B. \qquad (15)
\end{aligned}$$

Since they are both rational and knowledgeable about polarization optics (which is 'common knowledge'), given the pay-off table (Table 1), each one of them will choose his/her initial state to be $|C\rangle$ so that $\theta_A = \theta_B = 0$ and opt for the $\hat{D}$ strategy so that $P_{DD} = 1$ and the rest of the probabilities are zero. This will make the Nash Equilibrium DD the only possible outcome. This corresponds to the players playing 'pure strategies' in classical games.

Notice that Alice and Bob are not restricted to the choices $\hat{C}$ and $\hat{D}$ only. They can also choose a strategy $\hat{L} = \sigma_2$. It is clear from the pay-off Table 2 that $\hat{L}$ is the dominant strategy. Hence, a new Nash Equilibrium emerges which is LL. The pay-offs are now given by

$$\$_A = rP_{CC} + pP_{DD} + tP_{DC} + sP_{CD} + lP_{LL}, \qquad (16)$$

$$\$_B = rP_{CC} + pP_{DD} + sP_{DC} + tP_{CD} + lP_{LL}. \qquad (17)$$

This is an extended Prisoners' Dilemma game with three strategies [20]. Thus, in addition to 'cooperate' and 'defect', the players can also 'abstain', obtaining the 'loner's pay-off' L.

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Table 2: Extended Pay-off Table

	Bob: C	**Bob: D**	**Bob: L**
Alice: C	(3,3)	(0,5)	(2,2)
Alice: D	(5,0)	(1.1)	(2,2)
Alice: L	(2,2)	(2,2)	(2,2)

Table 2. If l denotes the 'loner's' pay-off, the condition $t > r > l > p > s$ is satisfied in this table [20].

The value of L must be set such that: (a) it is not greater than R, otherwise the advantage of not playing will be sufficiently large to ensure that players will always abstain, and (b) it is greater than S, otherwise there are no benefits to abstaining. Thus, the values of L can be set in the range $[S, R]$. We choose $L = 2$. It turns out that LL is also Pareto Optimal. So, there is no dilemma any more.
What is interesting is that all this has been achieved with product states. It will therefore be very interesting to investigate what additional results, if any, can be obtained if entanglement is taken into account in COM.

3 Entanglement and the Prisoners' Dilemma

It is now well known that entanglement does occur in classical polarization optics though quantum entanglement may have certain advantages over classical entanglement [18]. Consider two identical monochromatic sources S_A and S_B emitting *coherent* trains of horizontally polarized classical light pulses for Alice and Bob. Then the initial state is $|C\rangle_A \otimes |C\rangle_B$. Let Alice apply the Hadamard gate $\hat{H}$ (eqn (9)) on her state. That will produce the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|C\rangle_A + |D\rangle_A\right] \otimes |C\rangle_B = \frac{1}{\sqrt{2}}\left[|C\rangle_A \otimes |C\rangle_B + |D\rangle_A \otimes |C\rangle_B\right]. \tag{18}$$

Let a polarization filter be placed in Alice's train of pulses that selects $|C\rangle_A$ and $|D\rangle_A$ randomly, each 50% of the time, and let Bob apply $\hat{L} = \sigma_1$ on his state only whenever he finds Alice's pulse to be $|D\rangle_A$. That will generate the maximally entangled state

$$|\Psi^+\rangle_0 = \frac{1}{\sqrt{2}}[|C\rangle_A \otimes |C\rangle_B + |D\rangle_A \otimes |D\rangle_B]. \tag{19}$$

Then

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}}[\hat{U}_A|C\rangle_A \otimes \hat{U}_B|C\rangle_B + \hat{U}_A|D\rangle_A \otimes \hat{U}_B|D\rangle_B]. \tag{20}$$

If Bob chooses $\hat{D}$, the best response for Alice is (a) $\hat{D}$ or (b) $\hat{L}$, as will be explained shortly. But, if Bob chooses $\hat{L}$, Alice's best response is $\hat{L}$. Similarly, if Alice chooses $\hat{D}$, Bob's best response is (a) $\hat{D}$ or (b) $\hat{L}$, but if Alice chooses $\hat{L}$, Bob's best response is $\hat{L}$. The reason is that in case (a) $|\Psi^+\rangle_0$ is preserved, and in case (b) the state changes to

$$|\Psi_f^-\rangle = \frac{e^{i\pi}}{\sqrt{2}}[|C\rangle_A \otimes |C\rangle_B - |D\rangle_A \otimes |D\rangle_B]. \tag{21}$$

Maximal entanglement is preserved in both cases. However, now DD is no longer a Nash Equilibrium– the new NE in dominant strategies is LL (Table 2).
Thus, when entanglement is present, a new possibility arises in the PD game without requiring extension to a third strategy. And the new equilibrium is Pareto Optimal, which resolves the dilemma.
It turns out that all the strategies preserve the 'degree of entanglement' measure called 'concurrence'. In our case a general tensor product state between Alice's and Bob's beams would be

$$|\Psi\rangle = \alpha_1|0\rangle_A \otimes |0\rangle_B + \alpha_2|0\rangle_A \otimes |1\rangle_B + \alpha_3|1\rangle_A \otimes |0\rangle_B + \alpha_4|1\rangle_A \otimes |1\rangle_B \tag{22}$$

with $\sum_i |\alpha_i|^2 = 1$. Following Ref. [24], one can write the state as

$$|\Psi\rangle = c|\Psi_e\rangle + \sqrt{1-c^2}e^{i\chi}|\Psi_f\rangle \tag{23}$$

where $|\Psi_e\rangle$ is a maximally entangled state, $|\Psi_f\rangle$ is a factorizable state, and $c^2 = 2|\alpha_1\alpha_4 - \alpha_2\alpha_3|$ is the concurrence measure with $c \in [0,1]$. It is straightforward to verify that c is preserved by all the three strategies. In fact, this follows generally from the fact that c is invariant under local unitary transformations, and all the strategies are unitary operators.
Here, however, we would like to maintain caution in claiming that the Pareto Superior equilibrium obtained by (LL) is a cooperation strategy in the standard sense, since both players refrain from playing at that point, which is nevertheless a Pareto improvement for sure.

Superposition of strategies and FTP
Till now we have considered a pure strategy game, allowing for entanglement as described above. However, it has been demonstrated [12] that rational players do not follow standard rules of the game when contexts change, for example from full information to uncertainty. In our case when Alice and Bob face uncertainty (which may be either related to incomplete knowledge of strategy profiles of players or the criterion of rationality of each other's context), as contrasted to a full information scenario. Hence, under uncertainty (in generation of an action potential by a neuron [26]) the 'mental states' of players, i.e. their beliefs about strategy states of the other players, may be described by a superposition of states. For example, for two possible choices $|0\rangle'_A$ or $|1\rangle'_A$ available to Alice, Bob's state before a real move would be

$$\begin{aligned} |\phi\rangle_B &= \cos\chi|0\rangle'_B + e^{i\phi}\sin\chi|1\rangle'_B \\ &:= \alpha|0\rangle'_B + \beta|1\rangle'_B \end{aligned} \tag{24}$$

We assume that the two states $|0\rangle'$ and $|1\rangle'$ are orthogonal and represent the quiescent and firing bases. The amplitudes $\alpha = \cos\chi$ and $\beta = e^{i\phi}\sin\chi$ represent potentialities for a neuron to be quiescent or firing. This is a pure state which is fundamentally different from an ensemble of states. A pure state captures uncertainty. It can also be represented by a density matrix $\rho_B = |\phi\rangle_B\langle\phi|_B$. Since this particular pure state of Bob corresponds to Alice's states $|0\rangle'_A$ or $|1\rangle'_A$, Alice's state is either

$$|\phi_0\rangle_A = \alpha|0\rangle'_A \otimes |0\rangle'_B + \beta|0\rangle'_A \otimes |1\rangle'_B, \tag{25}$$

which is a product state, or

$$|\phi_1\rangle_A = \alpha|1\rangle'_A \otimes |0\rangle'_B + \beta|1\rangle'_A \otimes |1\rangle'_B \tag{26}$$

which is also a product state.
Now, since in general it is possible to define a 'mental state' by a density matrix, let us define a general mental state by

$$\Theta = \sum_{i,j} x_{i,j}|\phi_i\rangle\langle\phi_j|, \tag{27}$$

where (i, j) have values 0 and 1 [26]. Hence the initial mental states of Alice are

$$\begin{aligned} \Theta_{0A} &= |\phi_0\rangle_A\langle\phi_0|_A, && (28)\\ \text{or } \Theta_{1A} &= |\phi_1\rangle_A\langle\phi_1|_A && (29) \end{aligned}$$

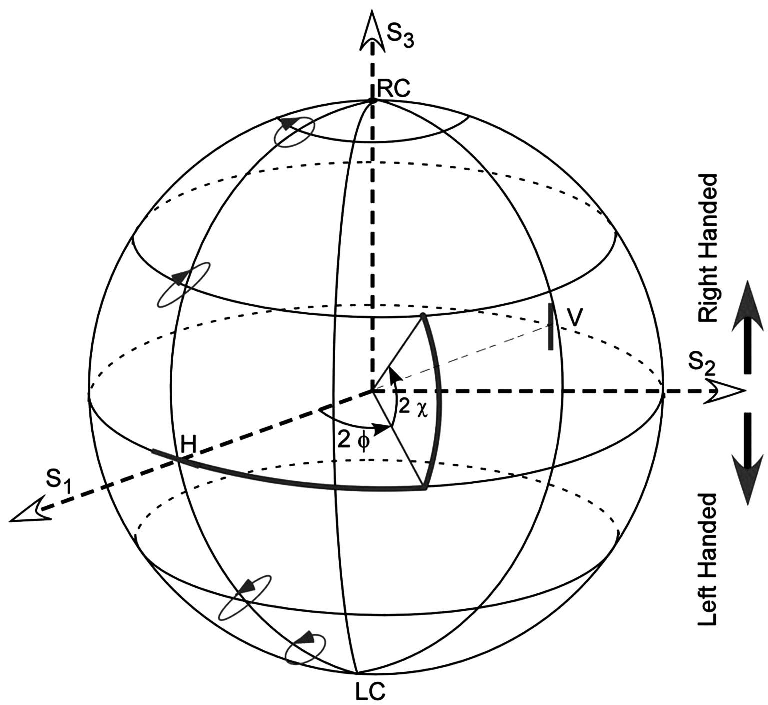
with $|\phi_0\rangle_A$ and $|\phi_1\rangle_A$ given by (25, 26). These mental states would evolve over time in a non-Markovian way, which would be the dynamics of the model. One can measure the probabilities of Alice choosing the states (25) and (26) as

$$\begin{aligned} P_{0A} &= \mathrm{Tr}\Theta_{0A} = \mathrm{Tr}|\phi_0\rangle_A\langle\phi_0|_A, && (30)\\ P_{1A} &= \mathrm{Tr}\Theta_{1A} = \mathrm{Tr}|\phi_1\rangle_A\langle\phi_1|_A. && (31) \end{aligned}$$

Such total probabilities contain interference terms due to the presence of superpositions embedded in them, and thus will differ significantly from classical probability-based computations. This is where the formula for total probability (FTP) developed in [10] comes in, as we will show in detail elsewhere.
Now let us consider the psychological process that reduces a pure state (a state of uncertainty) to a mixed state. In quantum mechanics if an observable $A = \sum_i a_i P_i$ with discrete eigenvalues a_i and projectors $P_i = |i\rangle\langle i|$ is measured on a system in a state ρ, then Lüders suggested the following rule or *ansatz* [27] for the consequent change of state:

$$\rho \to \rho'_k = \frac{P_k \rho P_k}{\mathrm{Tr} P_k \rho} \tag{32}$$

on the condition that the result a_k was obtained. This rule describes a change from a state ρ to a specific pure state ρ'_k in Hilbert space ($\rho \to \rho'_k, \rho'^2_k = \rho'_k$), and can be generalized to any Hilbert space theory, including classical polarization optics and COM, and to psychology as formulated in Ref. [26]. As regards the state of the ensemble after the



Sphere.png

Figure 1: The Poincaré Sphere

Lüders processes have occurred, one obtains the pure state

$$\rho' = \sum_k \rho'_k = \sum_k \frac{P_k \rho P_k}{\mathrm{Tr} P_k \rho}. \tag{33}$$

This is different from the von Neumann projection

$$\hat{\rho} = \sum_k (\mathrm{Tr} P_k \rho) \rho'_k = \sum_k P_k \rho P_k. \tag{34}$$

which produces a mixed state. According to Lüders, this is the processs of 'measurement followed by aggregation' in which *no selection or reading of individual results is done*. It is a non-unitary process. The Lüders process (33), on the other hand, is unitary. This can be seen by describing the change in Bob's state (eqn 24) as a *rotation* of the state (which is a point $(2\chi, 2\phi)$ on the Poincaré sphere) by -2ϕ about axis S_3 followed by a rotation by -2χ about axis S_2 (see Fig.1), so that the rotated state is H $(|0\rangle')$. Note that the numerator in (32) is a projection which changes the norm of the state (as in von Neumann's prescription), and this change is compensated by the denominator to restore the norm. This makes the Lüders rule in this case equivalent to a unitary rotation (see eqn (10)). Hence,

$$|\phi_0\rangle_A^r = |0\rangle'_A \otimes |0\rangle'_B, \tag{35}$$
$$|\phi_1\rangle_A^r = |1\rangle'_A \otimes |0\rangle'_B, \tag{36}$$

and

$$\hat{\Theta}_A = p_{0A} |\phi_0\rangle_A^r \langle\phi_0|_A^r + p_{1A} |\phi_1\rangle_A^r \langle\phi_1|_A^r \tag{37}$$

with $p_{0A} + p_{1A} = 1$, p_{0A} and p_{1A} being the probabilities for Alice to be initially in the states $|0\rangle'_A$ and $|1\rangle'_A$ respectively.
It will be readily seen that various combinations of rotations given by eqn (10) can be used to describe any desired change of polarization state, a point on the Poincaré sphere, to any other polarization state. Hence, such rotations can be used to describe all possible Lüders processes in COM which are the most appropriate to describe mental state changes from uncertainty to certainty.

4 Discussions and Conclusions

Distinguishing between correlated equilibrium games and entangled games, Aumann [21] initiated a strong literature on correlated equilibrium for classical games. Correlated equilibria always exist, and are achieved via a third party or a mediator. A trusted mediator can select strategies from joint probability distributions and 'privately' send such information to players, who then, based on such information, can maximize their payoffs, resulting in a Pareto superior equilibrium relative to standard Nash Equilibrium games. However, the challenge has been to establish such superior outcomes without the intervention of any mediator or, in effect, any communication between players, which is the

novelty of entangled games. Here the players can achieve such superior outcomes without the necessity to communicate.
In 'quantum correlated' games [22] players with full classical information can have access to a quantum entangled state, i.e. entangled qubits that are disjoint or spatially separated, and perform local operations on their qubits. No interaction is needed between the players. The novelty of our COM framework here is that similar outcomes can be accomplished utilizing *classical optical devices*, without any intervention of a mediator.
The quantum Prisoner's Dilemma has grown into an interesting exploration in itself [15]. Some authors [23] have suggested that any quantum game (where players have access to quantum strategies such as, for example, a super cooperation strategy as in [15], through quantum devices like quantum GATES) can be viewed as state preparation, transition and general measurement of the outcome state.
Pheonix et al [23] also envisage quantum games as *reverse engineered*, as for example, by choosing the optimal state first and then designing a playable game. However, general measurements of the outcome state can be fundamentally different from the classical game case, since in classical games the final output state (equilibrium state) and the pay-offs converge, while if a quantum game is designed via suitable instruments (i.e. it is playable), then there is no guarantee that the output state when measured would be an eigenstate of the measurement operator. Hence, in effect, the output state informs about an expectation of pay-offs.
However, since maximal quantum entanglement, which is a necessary condition for superior cooperation outcomes, is fragile in nature due to decoherence, overall we may conjecture that compared to any classical PD game, or more broadly a non-zero sum game of cooperation, one may achieve a superior equilibrium outcome if and only if entanglement is guaranteed to be stable in the presence of suitable operator representations of strategies.
The advantage of COM over pure quantum game modelling has already been pointed by us in the introduction section. Here we can further observe that in classical game theory in a single-shot game with no further rounds, the pay-off matrix is given, and hence empirically joint probabilities which help forming the expected pay-offs can either be computed by observing a large ensemble of PDs played by identical players, or in a repeated game scenario, by using Bayesian updating. However, in our formulation the probabilities would be computed by using the 'square of the amplitude rule', according to foundational theorems in Hilbert space theory, like Gleason's theorem [25]. Only Born's rule provides the unique formula for computing probabilities, hence for repeated games in this formulation we would continue using the same.
What we have shown is that classical entanglement, which is a well-accepted feature of classical optics and is stable, can be utilized instead of quantum entanglement to design such a game. While this will suffice for most cases, it is possible that quantum entanglement will have superiority in special cases, the quintessentially quantum ones.

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